Lesson 31 - MA 166 LEC 200

Announcements:
Exam 3 on Wed Apr 20 @ 6:30pm

11.2 & 11.4:

Power Series - Part 2

☆ Warm Up: Evaluate the Limit

$$\lim_{x \to 0} \frac{\sin(2x) - 2x}{x^3} = ?$$

(a) $\frac{8}{3}$

(b) 2

(c) 0

(d) $-\frac{2}{3!}$

$\boxed{\text{(e)} \ -\frac{4}{3}}$

Taylor series $\sin(2x) = \cancel{2x} - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \ldots$

$$\lim_{x \to 0} \frac{\sin(2x) - 2x)}{x^3} = \frac{-\frac{2^3 \cancel{x^3}}{3!} + \frac{2^5 x^{\cancel{5}\,2}}{5!}}{\cancel{x^3}}$$

$$= -\frac{2^3}{3!} = -\frac{8}{6} = \boxed{-\frac{4}{3}}$$

I. Differentiating & Integrating Power Series

Given $\sum_{k=0}^{\infty} c_k x^k$

want to find

(1) $\frac{d}{dx}\left(\sum_{k=0}^{\infty} c_k x^k\right)$

(2) $\int \left(\sum_{k=0}^{\infty} c_k x^k\right) dx$

<u>Principle</u>: Go term by term

<u>Ex(1)</u>: $f(x) = \dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

$$= \sum_{k=0}^{\infty} x^k$$

$R = 1 \quad I = (-1, 1)$

move inside

$$f'(x) = \frac{-1}{(1-x)^2} \cdot (-1) = \frac{1}{(1-x)^2} = \frac{d}{dx}\left(\sum_{k=0}^{\infty} x^k\right)$$

$$= \sum_{k=0}^{\infty} \frac{d}{dx}(x^k) = \sum_{k=0}^{\infty} k x^{k-1}$$

Q: What is the radius & interval of convergence for $f'(x)$?

<u>Claim</u>: since $f(x)$ converges on $(-1,1)$ then $f'(x)$ also converges on $(-1,1)$

Theorem 11.5 in textbook

<u>BUT</u>: we don't know about the endpoints $x=-1$, and $x=1$

<u>Check</u>: @ $x=-1$

$$\sum_{k=0}^{\infty} \underbrace{k(-1)^{k-1}}_{a_k}$$

Alternating → A.S.T

$\lim_{k\to\infty} a_k \stackrel{?}{=} 0$

$\lim_{k\to\infty} k = \infty$

by A.S.T. series diverges

@ $x=1$ $\sum_{k=0}^{\infty} k(1)^{k-1} = \sum_{k=0}^{\infty} k$ diverges by Divergence Test

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} kx^{k-1} \quad (k=1)$$

$R = 1$

$I = (-1, 1)$

(starting point should be $k=1$

$\frac{d}{dx} \sum x^k$

when $k=0$

$\frac{d}{dx}(1) = 0$

$= 0 + 1 \cdot + 2x + 3x^2 + \cdots$)

Warning: In this case

$I = (-1, 1)$ for $f(x)$

$I = (-1, 1)$ for $f'(x)$

happen to be the same

$\longrightarrow$ NOT always true

need to check endpoints

Ex(2): $f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

Find the power series for $\int f(x)\,dx$

$$\int f(x)\,dx = \int \frac{1}{1-x}\,dx = -\ln|1-x| + C$$

pull inside

$\int f(x)dx = \int \frac{1}{1-x} dx$

pull inside

$$\int \sum_{k=0}^{\infty} x^k \, dx = \sum_{k=0}^{\infty} \int x^k \, dx = \left(\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \right) + C$$

defined at $k=0$ ✓

Theorem 11.5: $R=1$ for $f(x)$, so $R=1$ for $\int f(x)dx$

But, we need to check the endpoints

Check:

@ $x=-1$ $\quad C + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+1}$

Alternating

$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k+1} = 0$

a_k is nonincreasing

by the A.S.T. this series converges

@ $x=+1$ $\quad C + \sum_{k=0}^{\infty} \frac{(1)^{k+1}}{k+1} = C + \sum_{k=0}^{\infty} \frac{1}{k+1}$

let $j=k+1$

$$= C + \sum_{j=1}^{\infty} \frac{1}{j}$$

Harmonic series

p-series $p=1$

diverges

$R=1 \qquad I=[-1,1)$

NOTE: $-\ln|1-x| = C + \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$

what is C? plug in $x=0$ ... $\frac{0}{k+1}$

—

what is C? plug in $x=0$

$$-\ln|1-0| = 0 = C + \cancel{\sum_{k=0}^{\infty} \frac{0^{k+1}}{k+1}}^{\,0} = 0$$

$$\boxed{C=0}$$

$$\boxed{-\ln|1-x| = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}}$$

Summary: $f(x) = \displaystyle\sum_{k=0}^{\infty} c_k (x-a)^k$

Thm 11.5 $\qquad R,\ I$

1. Differentiation

$$f'(x) = \sum_{k=1}^{\infty} c_k \cdot k\,(x-a)^{k-1}$$

- R remains the same
- I may change at the endpoints

2. Integration

$$\int f(x)\,dx = C + \sum_{k=0}^{\infty} c_k \frac{(x-a)^{k+1}}{k+1}$$

- R remains the same
- I may change at the endpoints

II. Functions to Power Series

power series representation

Ex (3): Find the power series representation of $\tan^{-1}(x)$

Set $f(x) = \tan^{-1}(x)$

then $f'(x) = \dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)}$

we know $\dfrac{1}{1-r} = \sum_{k=0}^{\infty} r^k$ let $r = -x^2$

$$f'(x) = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

$R = 1$ $I = (-1, 1)$

$$f(x) = \tan^{-1}(x) = \int f'(x)\,dx = \int \sum_{k=0}^{\infty} (-1)^k x^{2k}\,dx$$

$$= \sum_{k=0}^{\infty} \int (-1)^k x^{2k}\,dx = \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \right) + C$$

Q: What is the value of C?

$$\tan^{-1}(x) = C + \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

Evaluate at $x = 0$

$$\tan^{-1}(0) = 0 = C + \sum_{k=0}^{\infty} (-1)^k \cancelto{0}{\frac{0^{2k+1}}{2k+1}}$$

$$\boxed{C = 0}$$

Check convergence @ $x=-1$, and $x=+1$

...

III. Power Series to Functions:

Ex: $\displaystyle\sum_{k=0}^{\infty} \frac{(1-2x)^k}{k!}$ what function $f(x)$ does this represent?

Know that $\displaystyle e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!}$ let $y = 1-2x$

$$\sum_{k=0}^{\infty} \frac{(1-2x)^k}{k!} = e^y = e^{1-2x}$$

Ex: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{4^k} k x^{2k} = f(x)$?

$$= \sum_{k=1}^{\infty} k \left(\frac{-x^2}{4}\right)^k$$

$$\frac{1}{1-y} = \sum_{k=0}^{\infty} y^k \quad \longleftarrow \text{Differentiate}$$

$$\left(\frac{1}{(1-y)^2} = \left(\frac{1}{1-y}\right)' = \sum_{k=1}^{\infty} k y^{k-1}\right) y$$

$$\frac{y}{(1-y)^2} = y \sum_{k=1}^{\infty} k y^{k-1} = \sum_{k=1}^{\infty} k y y^{k-1}$$

$$= \sum^{\infty} k y^k$$

$$= \sum_{k=1}^{\infty} k y^k$$

Let $y = \frac{-x^2}{4}$

$$\sum_{k=1}^{\infty} k \left(\frac{-x^2}{4}\right)^k = \frac{y}{(1-y)^2} = \frac{(-x^2/4)}{\left[1-\left(\frac{-x^2}{4}\right)\right]^2}$$

$$= \frac{-4x^2}{[4+x^2]^2}$$