Lesson 31 - MA 166 LEC 286

Announcements:
Exam 3 on Wed Apr 20 @ 6:30pm

11.2 & 11.4:

Power Series - Part 2

☆ Warm Up: Evaluate the Limit

$$\lim_{x\to 0} \frac{\sin(2x)-2x}{x^3} = ?$$

(a) $\frac{8}{3}$

(b) 2

(c) 0

(d) $-\frac{2}{3!}$

$\boxed{(e)\ -\frac{4}{3}}$

Maclaurin series $\sin(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \dots$

Let $y=2x$ $\quad \sin(2x) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \dots$

$$\lim_{x\to 0} \frac{\sin(2x)-2x}{x^3} = \lim_{x\to 0} \frac{\left(\cancel{2x} - \frac{2^3x^3}{3!} + \frac{2^5x^5}{5!} + \dots\right) - \cancel{2x}}{x^3}$$

$$= \lim_{x\to 0} \frac{-\frac{2^3\cancel{x^3}}{3!} + \frac{2^5\cancel{x^5}^{\,x^2}}{5!} + \dots}{\cancel{x^3}} = \lim_{x\to 0} -\frac{2^3}{3!} + \cancel{\frac{2^5x^2}{5!}}^{\to 0} + \dots^{\,0}$$

$$= -\frac{2^3}{3!} = -\frac{8}{6} = \boxed{-\frac{4}{3}}$$

I. Differentiating & Integrating Power Series:

Given $\sum_{k=0}^{\infty} c_k x^k$

Want to Find:

Want to find:

(1) $\frac{d}{dx}\left(\sum_{k=0}^{\infty} c_k x^k\right)$ (2) $\int\left(\sum_{k=0}^{\infty} c_k x^k\right) dx$

<u>Principle</u>: go term by term

<u>Ex(1)</u>: $f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ $R = 1$

$I = (-1, 1)$

$$f'(x) = \left(\frac{1}{1-x}\right)' = \frac{-1}{(1-x)^2}\cdot(-1) = \frac{1}{(1-x)^2}$$

move inside

$$= \frac{d}{dx}\left(\sum_{k=0}^{\infty} x^k\right) \overset{\text{WANT}}{=} \sum_{k=0}^{\infty} \frac{d}{dx}\left(x^k\right)$$

BE CAREFUL — at the $k=0$ term

$$= \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots\right)$$

$$= \frac{d}{dx}\left[1 + \sum_{k=1}^{\infty} x^k\right]$$

$$= \cancel{\frac{d}{dx}(1)}^{\,0} + \sum_{k=1}^{\infty} \frac{d}{dx}\left(x^k\right)$$

$$= \sum_{k=1}^{\infty} k x^{k-1} = f'(x) = \frac{1}{(1-x)^2}$$

<u>Claim</u> (Theorem 11.5 in Textbook)

If $f(x)$ has R, then $f'(x)$ also has R as radius of conv.

BUT, they may be different at endpoints.

$f(x)$ has $R=1$, $\Rightarrow$ $f'(x)$ has $R=1$

Need to check $x=-1$ and $x=+1$

Check: @ $x=-1$ $\displaystyle\sum_{k=1}^{\infty} \underbrace{k(-1)^{k-1}}_{a_k}$ Alternating A.S.T

$\lim_{k\to\infty} a_k = \lim_{k\to\infty} k = \infty$

diverges

@ $x=1$ $\displaystyle\sum_{k=1}^{\infty} k(+1)^{k-1} = \sum_{k=1}^{\infty} k$ Divergence Test

$\lim_{k\to\infty} a_k$

$= \lim_{k\to\infty} k = \infty$

diverges

$$f'(x) = \sum_{k=1}^{\infty} k x^{k-1}$$ $R=1$

$I = (-1,1)$

☆ Warning: in this example $I=(-1,1)$ for both $f(x)$ and $f'(x)$

This is NOT always true

Ex(2): $f(x) = \dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k$

Find the power series for $\int f(x)\,dx$

$$\int f(x)\,dx = \int \frac{1}{1-x}\,dx = -\ln|1-x|$$

move inside

$$\int \sum^{\infty} x^k \, d \quad = \sum^{\infty} \int x^k\,dx$$

move inside

$$= \int \sum_{k=0}^{\infty} x^k \, dx = \sum_{k=0}^{\infty} \int x^k \, dx$$

$$= \left(\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} \right) + C \quad = f'(x) = -\ln|1-x|$$

Theorem 11.5: If $f(x)$ has R as radius of convergence, then $\int f(x)dx$ also has radius of convergence R,

$f(x)$ has $R=1$, so $\int f(x)dx$ also $R=1$

Still need to check the endpoints:
check at $x=1$ and $x=-1$

@ $x=-1$ $\quad C + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+1}$ $\quad$ A.S.T $\lim_{k\to\infty} \frac{1}{k+1} = 0$

converges $\quad$ are nonincreasing

@ $x=+1$ $\quad C + \sum_{k=0}^{\infty} \frac{(1)^{k+1}}{k+1} = C + \sum_{k=0}^{\infty} \frac{1}{k+1} = \sum_{j=1}^{\infty} \frac{1}{j}$

let $j = k+1$

Harmonic series
p-series w/ $p=1$
diverges

$$\boxed{R=1, \quad I=[-1,1)}$$

NOTE: What is the value of C?

$$-\ln|1-x| = C + \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

plug in any value of x in $I=[-1,1)$
choose $x=0$

plug in any value of x in $I = (-1,1)$

choose $x = 0$

$$0 = -\ln|1-0| = C + \cancelto{0}{\sum_{k=0}^{\infty} \frac{0^{k+1}}{k+1}} = \boxed{C = 0}$$

$$\boxed{-\ln|1-x| = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}}$$

☆ Summary: $f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$ with R, I

Theorem 11.5

Then:
$$f'(x) = \sum_{k=1}^{\infty} c_k \cdot k (x-a)^{k-1}$$

$$\int f(x)\,dx = C + \sum_{k=0}^{\infty} c_k \frac{(x-a)^{k+1}}{k+1}$$

For both:

- R remains the same
- I may change at the endpoints

II. Functions to Power Series:

Ex (3): Find the power series representation of $\tan^{-1}(x)$

set $f(x) = \tan^{-1}(x)$

then $f'(x) = \dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)}$

$1 \qquad \sum^{\infty} r^k \qquad$ let $r = -x^2$

$$\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k \qquad \text{let } r = -x^2$$

$$f'(x) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

$$f(x) = \int f'(x)\,dx = \int \sum_{k=0}^{\infty} (-1)^k x^{2k}\,dx$$

$$= \sum_{k=0}^{\infty} \int (-1)^k x^{2k}\,dx = \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \right) + C$$

$$\tan^{-1}(x) = C + \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

plug in $x=0$

$$0 = \tan^{-1}(0) = C + \sum_{k=0}^{\infty} (-1)^k \cancel{\frac{0^{2k+1}}{2k+1}} \to 0$$

$$\boxed{C=0}$$

$$\boxed{\tan^{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}}$$

III. Power Series to Functions:

Ex: $\displaystyle\sum_{k=0}^{\infty} \frac{(1-2x)^k}{k!} = f(x)?$

Note that $\displaystyle e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!}$ let $y = 1-2x$

$1-2x$

Note that $e \;=\; \sum_{k=0} \frac{\quad}{k!}$

$$= \sum_{k=0}^{\infty} \frac{(1-2x)^k}{k!} = e^{1-2x}$$

Ex: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{4^k}\, k\, x^{2k} = f(x)?$

$$= \sum_{k=1}^{\infty} k\left(\frac{-x^2}{4}\right)^k \quad \sim \quad \sum k y^k$$

Geometric series $\qquad \dfrac{1}{1-y} = \displaystyle\sum_{k=0}^{\infty} y^k$

$$\left[\frac{1}{(1-y)^2} = \left(\frac{1}{1-y}\right)' = \frac{d}{dx}\left(\qquad\right) = \sum_{k=1}^{\infty} k y^{k-1}\right] y$$

$$\frac{y}{(1-y)^2} = y\left(\sum_{k=1}^{\infty} k y^{k-1}\right) = \sum_{k=1}^{\infty} k y^k$$

Let $y = \dfrac{-x^2}{4}$

$$\sum_{k=1}^{\infty} k\left(\frac{-x^2}{4}\right)^k = \frac{\left(\frac{-x^2}{4}\right)}{\left[1 - \left(\frac{-x^2}{4}\right)\right]^2} = \frac{-4x^2}{\left[4+x^2\right]^2}$$