12.2: Polar Coordinates (Basics)

Announcements:

- MyLab Math outage this morning
- will give an extension on Hw 31 and Quiz 10

☆Warm Up:

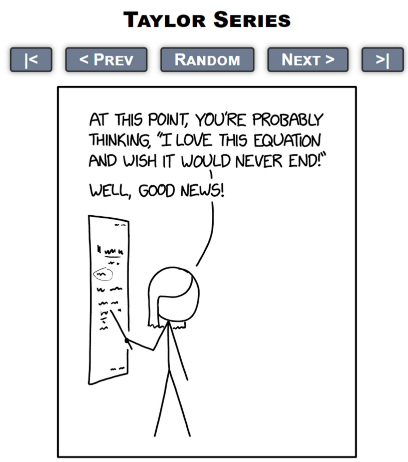
Find the power series of $\int e^{-x^2}\,dx$

(a) $\boxed{C + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{x^{2k+1}}{(2k+1)}}$

(b) $C + \sum_{k=0}^{\infty} \frac{1}{k!} \frac{x^{k+3}}{(k+3)}$

(c) $C + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} x^{2k+1}$

(d) $C + \sum_{k=0}^{\infty} \frac{1}{k!} \frac{x^{k+1}}{(k+1)}$



TAYLOR SERIES EXPANSION IS THE ~~WORST~~ BEST

XKCD

$$e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!} \qquad \text{let } y = -x^2$$

$$\int e^{-x^2}\,dx = \int \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!}\,dx = \int \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}\,dx$$

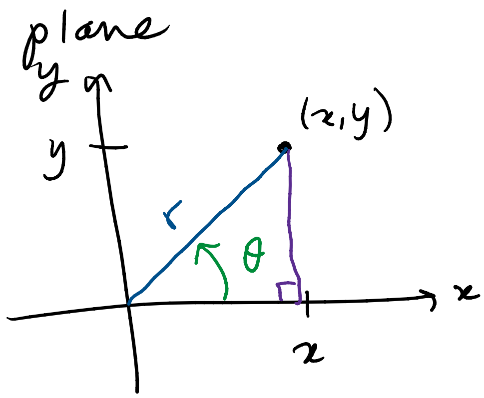
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int x^{2k}\,dx = \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{x^{2k+1}}{2k+1} \right) + C$$

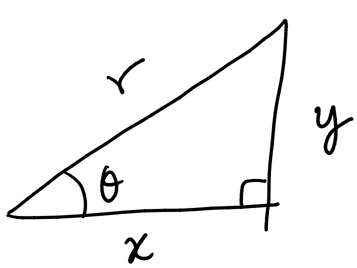
I. Polar Coordinates:

Given a point in the x-y plane

(x, y) – Cartesian coordinates

(r, θ) – Polar coordinates





Pythagorean Theorem

$r^2 = x^2 + y^2$

$r = \sqrt{x^2+y^2}$

Trig

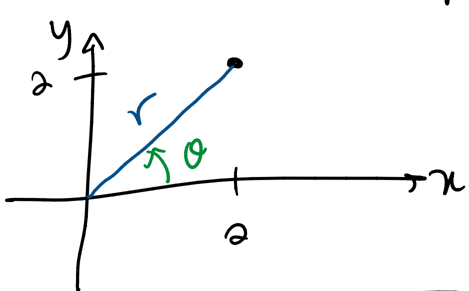
$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Converting Between coordinate systems

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases} \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Ex(1): $(x,y) = (2,2)$ plot and write it in polar coordinates



$r = \sqrt{x^2+y^2}$

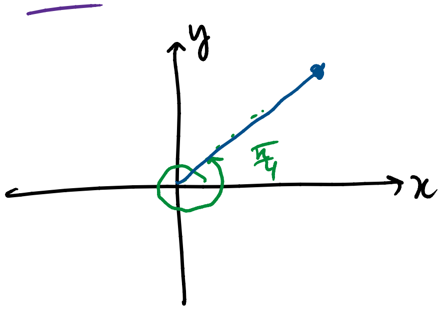
$= \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{2}\right)$

$= \tan^{-1}(1) = \frac{\pi}{4}$

$$\boxed{(r,\theta) = \left(2\sqrt{2}, \frac{\pi}{4}\right)}$$

Q: Plot $(r,\theta) = \left(2\sqrt{2}, \frac{9\pi}{4}\right)$



$r = 2\sqrt{2}$

$\theta = \frac{9\pi}{4} = 2\pi + \frac{\pi}{4}$

This is the same point

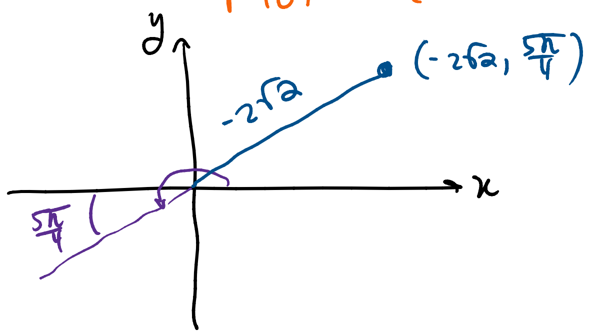
$\left(2\sqrt{2}, \frac{\pi}{4}\right)$ and $\left(2\sqrt{2}, \frac{9\pi}{4}\right)$ represent the same point

NOTE: r can be negative

plot $\left(-2\sqrt{2}, \frac{5\pi}{4}\right)$

NOTE: r can be negative

Plot $(-2\sqrt{2}, \frac{5\pi}{4})$



the same point again

$(2\sqrt{2}, \frac{\pi}{4})$

$(2\sqrt{2}, \frac{9\pi}{4})$

$(-2\sqrt{2}, \frac{5\pi}{4})$

Equivalent points in Polar coordinates

$$(r, \theta) \equiv (r, \theta + 2\pi) \equiv (-r, \theta + \pi)$$

all represent the same point

HOTSEAT: Which of the following points are equivalent

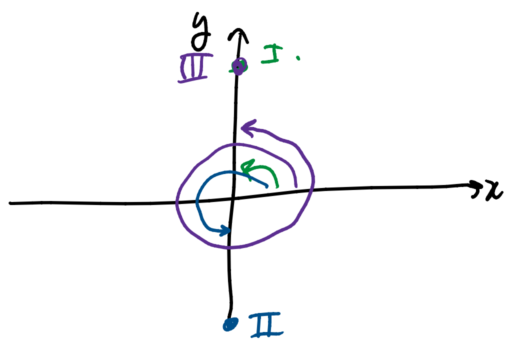
I. $(2, \frac{\pi}{2})$ II. $(2, \frac{3\pi}{2})$ III. $(2, \frac{5\pi}{2})$

(a) I, II, and III

(b) I and III only

(c) I and II only

(c) II and III only



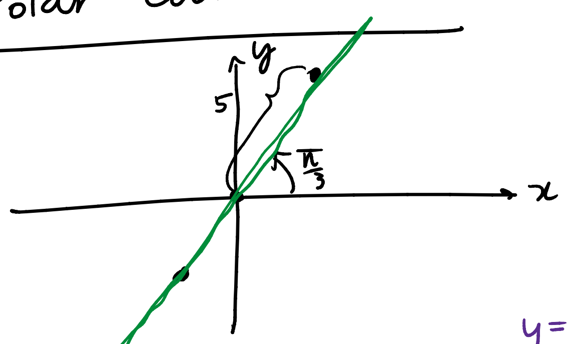
II Basic Curves in Polar Coordinates

① $\theta = \frac{\pi}{3}$ $\boxed{\text{line}}$

$r = 0$ $\theta = \frac{\pi}{3}$

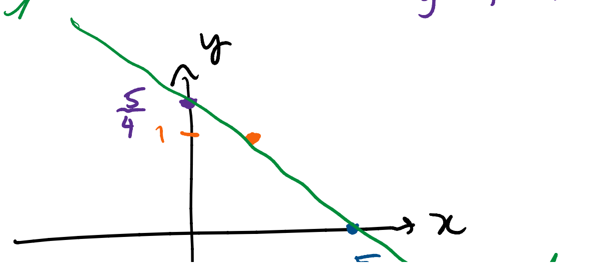
$r = 5$ $\theta = \frac{\pi}{3}$

$r = -3$ $\theta = \frac{\pi}{3}$

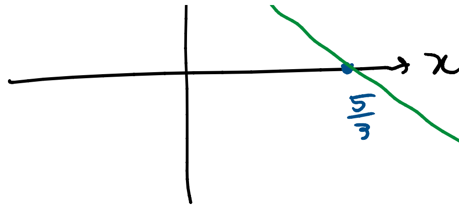


$y = f(x)$

② $r = \dfrac{5}{3\cos\theta + 4\sin\theta}$



② $r = \dfrac{\quad}{3\cos\theta + 4\sin\theta}$



line
$y = -\frac{3}{4}x + \frac{5}{4}$

$\theta = 0 \qquad r = \dfrac{5}{3\cos(0) + \cancel{4\sin 0}} = \dfrac{5}{3}$

$\theta = \frac{\pi}{2} \qquad r = \dfrac{5}{\cancel{3\cos(\frac{\pi}{2})} + 4\sin(\frac{\pi}{2})} = \dfrac{5}{4}$

$\theta = \frac{\pi}{3} \qquad r = \dfrac{5}{3\cos(\frac{\pi}{3}) + 4\sin(\frac{\pi}{3})} = \dfrac{5}{3(\frac{1}{2}) + 4(\frac{\sqrt{3}}{2})} = \dfrac{5}{\left(\frac{3+4\sqrt{3}}{2}\right)} = \dfrac{10}{3+4\sqrt{3}}$

~ 1.01

$$r = \frac{5}{3\cos\theta + 4\sin\theta}$$

$$r(3\cos\theta + 4\sin\theta) = 5$$

$$\underbrace{3r\cos\theta}_{x} + \underbrace{4r\sin\theta}_{y} = 5$$

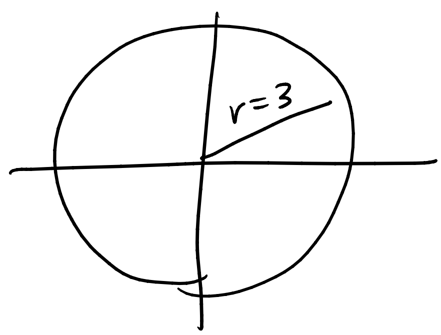
$$3x + 4y = 5 \longrightarrow \boxed{y = -\frac{3x}{4} + \frac{5}{4}}$$

③ $r = 3$

circle

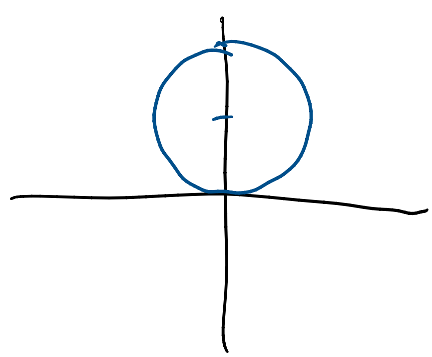
radius $= 3$

center $= (0,0)$



④ $(r = 2\sin\theta) \times r$

$$\underbrace{r^2}_{x^2+y^2} = 2\underbrace{r\sin\theta}_{y}$$



$$x^2 + y^2 = 2y$$

$$x^2 + y^2 - 2y + 1 = 0 + 1 \quad \text{circle}$$

$$x^2 + y^2 - 2y + 1 = 0 + 1$$
$$x^2 + (y-1)^2 = 1$$

circle
center $(0,1)$
radius 1

HOTSEAT: The equation $r = 10\cos\theta + 24\sin\theta$ represents a circle. Find the center (x,y) and radius r

(a) $r = 13$ $\quad (x,y) = (5,12)$

(b) $r = 13$ $\quad (x,y) = (10,24)$

(c) $r = 169$ $\quad (x,y) = (10,12)$

$$r^2 = 10r\cos\theta + 24r\sin\theta$$
$$x^2 + y^2 = 10x + 24y$$
$$(x^2 - 10x + 5^2) + (y^2 - 24y + 12^2) = 5^2 + 12^2$$
$$(x-5)^2 + (y-12)^2 = 13^2$$

Equation of a circle
$$r = 2a\cos\theta + 2b\sin\theta$$
has radius $\sqrt{a^2+b^2}$
center (a,b)

III. Cartesian to Polar:

Given $y = x^2$ $\rightarrow$ convert to polar

$$\ldots = (r\cos\theta)^2 = r^2\cos^2\theta$$

Given:

$$(r\sin\theta) = (r\cos\theta)^2 = r^2\cos^2\theta$$

Want: $r = f(\theta)$

$$\frac{r^{\cancel{2}}\,\cancel{\cos^2\theta}}{\cancel{r}\,\cancel{\cos^2\theta}} = \frac{\cancel{r}\sin\theta}{\cancel{r}\cos^2\theta}$$

$$r = \frac{\sin\theta}{\cos^2\theta} = \left(\frac{\sin\theta}{\cos\theta}\right)\frac{1}{\cos\theta}$$

$$\boxed{r = \tan\theta\sec\theta}$$