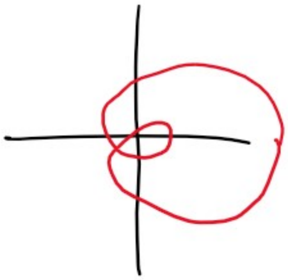
Lesson 34 - MA 166 LEC 200

12.3: Calculus in Polar Coordinates – Part 2

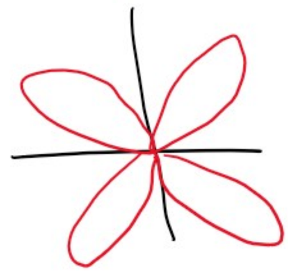
Announcements:
- Final Exam Mon May 2nd @ 10:30am
  - ↳ study guide
- Exam 3 Benchmark posted on Wed

Warm Up:
Which of the following is the graph of $r = 4\sin(3\theta)$?

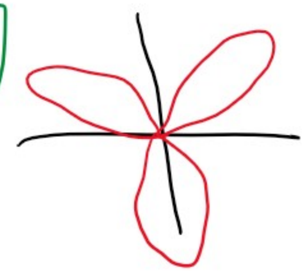
(a)



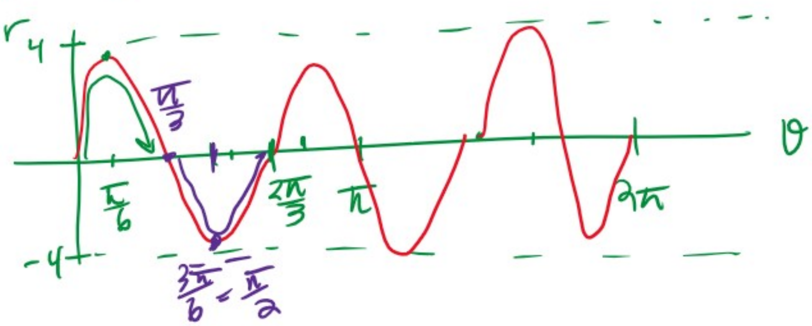
(b)

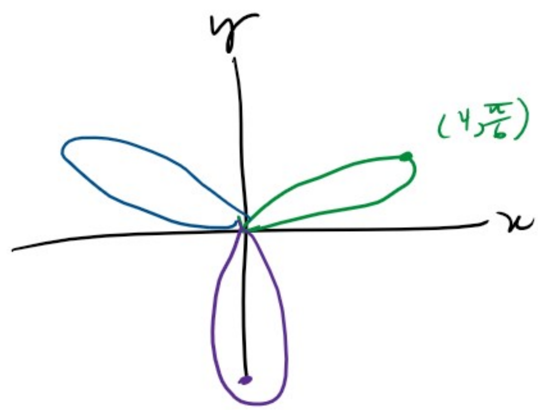


(c)



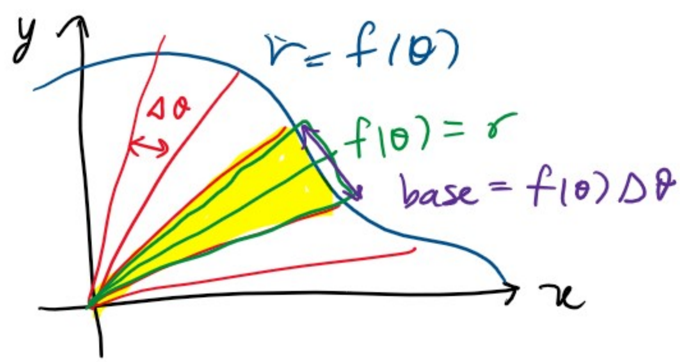
$r = 4\sin(3\theta)$





I. Areas of Regions bounded by Polar Curves

Idea: slice + sum
- slice in θ



area of wedge ≈ triangle

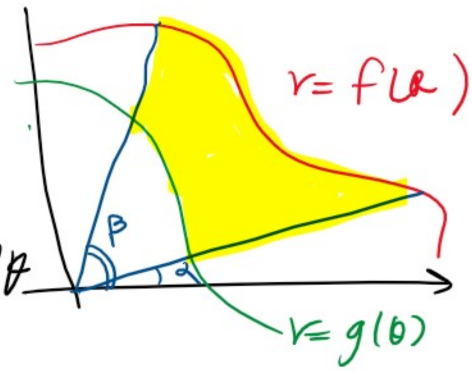
$= \frac{1}{2}$ base × height

$= \frac{1}{2}\left(f(\theta)\Delta\theta\right)\left(f(\theta)\right) = \frac{1}{2}\left[f(\theta)\right]^2 \Delta\theta$

$\sum \frac{1}{2}\left[f(\theta)\right]^2 \Delta\theta \longrightarrow$

$$\text{Area} = \sum \frac{1}{2}\left[f(\theta)\right]^2 \Delta\theta \xrightarrow[\text{as } \Delta\theta \to 0]{\text{limit}}$$

$$\boxed{\text{Area} = \int_\alpha^\beta \frac{1}{2}\left[f(\theta)\right]^2 d\theta}$$

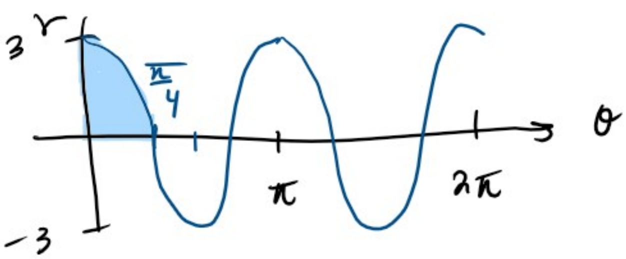


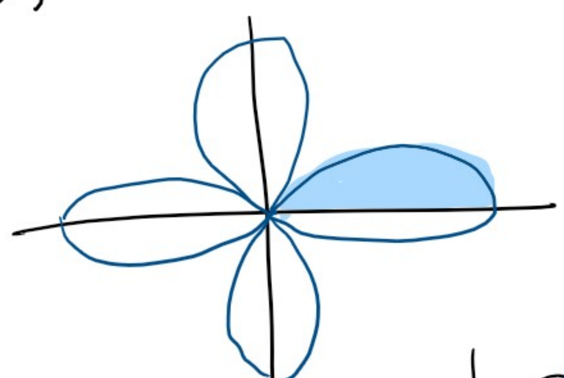
__Two curves:__

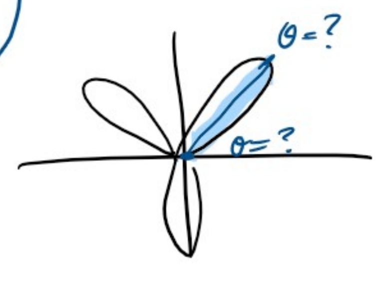
$$\text{Area} = \int_\alpha^\beta \frac{1}{2}\left[\left(f(\theta)\right)^2 - \left(g(\theta)\right)^2\right] d\theta$$

__Ex(1)__: Find the area of the four-leaf rose

$$r = f(\theta) = 3\cos(2\theta)$$







Area = 8 × 

$$= 8\int_0^{\frac{\pi}{4}} \frac{1}{2}\left[f(\theta)\right]^2 d\theta$$

$$= \frac{8}{2}\int_0^{\frac{\pi}{4}} \left[3\cos(2\theta)\right]^2 d\theta$$

$$\cos^2\varphi = \frac{1+\cos(2\varphi)}{2}$$

$$\varphi = 2\theta$$

$$= 9\int^{\frac{\pi}{4}} \cos^2(2\theta)\, d\theta$$

$$= 4 \cdot 9 \int_0^{\frac{\pi}{4}} \cos^2(2\theta)\, d\theta$$

$\varphi = 2\theta$

$\dfrac{1 + \cos(4\theta)}{2}$

$$= 36 \int_0^{\frac{\pi}{4}} \frac{1 + \cos(4\theta)}{2}\, d\theta$$

$$= 18 \int_0^{\frac{\pi}{4}} 1 + \cos(4\theta)\, d\theta$$

$$= 18 \left[\theta + \frac{\sin(4\theta)}{4} \right]_0^{\frac{\pi}{4}}$$

$$= 18 \left[\frac{\pi}{4} + \cancel{\frac{\sin\left(4 \cdot \frac{\pi}{4}\right)}{4}}^{\,0} - 0 - \cancel{\frac{\sin(0)}{4}}^{\,0} \right]$$

$$= \frac{18\pi}{4} = \boxed{\frac{9\pi}{2}}$$

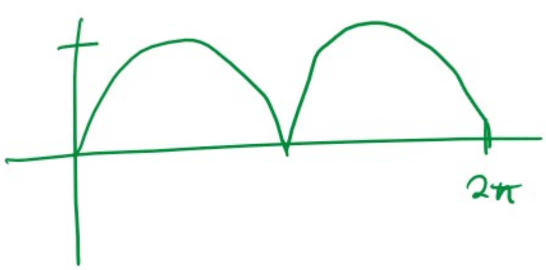
HOTSEAT: Find the area inside the curve $r = \sin^2\theta$ (ignore the 5 on Hot Seat)

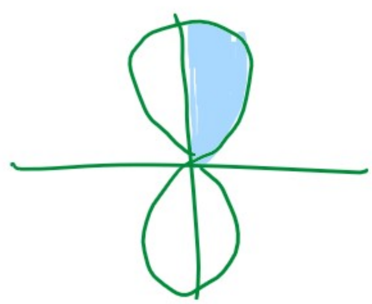
(a) $\frac{\pi}{2}$

(b) $\frac{3\pi}{8}$

(c) 2π

(d) $\frac{3\pi}{4}$





$$A = \frac{1}{2} \int_\alpha^\beta [f(\theta)]^2\, d\theta = \frac{1}{2} \int_0^{2\pi} [\sin^2\theta]^2\, d\theta$$

$$= \frac{1}{2} \int^{2\pi} \left[\frac{1 - \cos(2\theta)}{2} \right]^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left[\frac{1-\cos(2\theta)}{2}\right]^2 d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{4}\int_0^{2\pi} 1 - 2\cos(2\theta) + \cos^2(2\theta)\, d\theta$$

$$= \frac{1}{8}\int_0^{2\pi} 1 - 2\cos(2\theta) + \left(\frac{1+\cos(4\theta)}{2}\right) d\theta$$

$$= \frac{1}{8}\int_0^{2\pi} \frac{3}{2} - 2\cos(2\theta) + \frac{\cos(4\theta)}{2}\, d\theta$$

$$= \frac{1}{8}\left[\frac{3\theta}{2} - \frac{2\sin(2\theta)}{2} + \frac{\sin(4\theta)}{2\cdot 4}\right]_0^{2\pi}$$

$$= \frac{1}{8}\left[\frac{3\cdot\cancel{2}\pi}{\cancel{2}} - \cancelto{0}{\frac{2\sin(4\pi)}{2}} + \cancelto{0}{\frac{\sin(4\cdot 2\pi)}{8}} - 0 + \cancelto{0}{\frac{2\sin(0)}{2}} - \cancelto{0}{\frac{\sin(0)}{8}}\right]$$

$$= \boxed{\frac{3\pi}{8}}$$

II. Arc Length:

The arc length L of the polar curve $r = f(\theta)$ on $[\alpha, \beta]$ is

$$L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

$$L = \int_{\alpha}^{\beta} \sqrt{(f(\theta))^2 + [f'(\theta)]^2}\ d\theta$$

Ex (2): Find the arc length of cardiod $r = 1 + \cos\theta$ on $0 \le \theta \le 2\pi$

$f(\theta) = 1 + \cos\theta \qquad f'(\theta) = -\sin\theta$

$$L = \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\ d\theta$$

$$= \int_0^{2\pi} \sqrt{1 + 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{=1}}\ d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\ d\theta$$

Half angle formula

$4\left(\cos^2\theta = \frac{1+\cos(2\theta)}{2}\right)$

$4\cos^2\theta = 2 + 2\cos(2\theta)$

$\theta = \frac{\theta}{2}$, $2\theta = \theta$

$$= \int_0^{2\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\ d\theta$$

$$= 2\int_0^{2\pi} \left|\cos\left(\frac{\theta}{2}\right)\right| d\theta$$

$$\cancel{= 2\left[\frac{\sin\left(\frac{\theta}{2}\right)}{\frac{1}{2}}\right]_0^{2\pi} = 4\left[\sin\left(\frac{2\pi}{2}\right) - \sin\left(\frac{0}{2}\right)\right.}$$

$\left[\int^{\pi} \ldots (\theta)\ d\theta \quad \ldots \int^{2\pi} \left(-\cos\left(\frac{\theta}{2}\right)\right) d\theta\right]$

$$= 2\left[\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta + \int_{\pi}^{2\pi} \left(-\cos\left(\frac{\theta}{2}\right)\right) d\theta\right]$$

$$= 2\left[2\sin\left(\frac{\theta}{2}\right)\Big|_0^{\pi} + \left(-2\sin\left(\frac{\theta}{2}\right)\Big|_{\pi}^{2\pi}\right)\right]$$

$$= 2\left[2 + 2\right] = 8$$