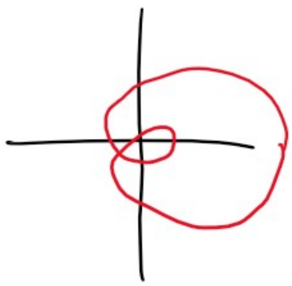
Lesson 34 - MA 166 LEC 286

12.3: Calculus in Polar Coordinates – Part 2

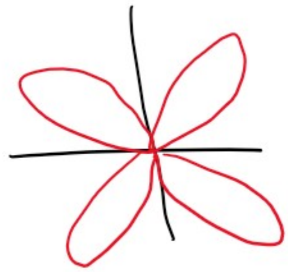
Announcements:
- Final Exam Mon May 2nd @ 10:30am
  - ↪ study guide
- Exam 3 Benchmark posted on Wed

Warm Up:
Which of the following is the graph of $r = 4\sin(3\theta)$?

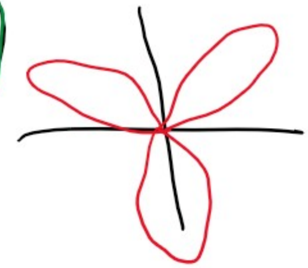
(a)



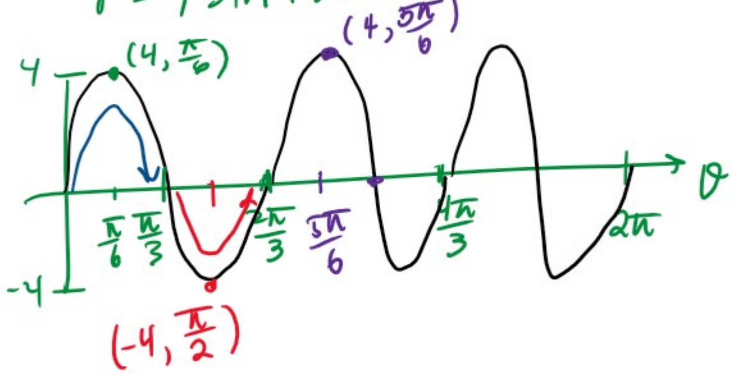
(b)

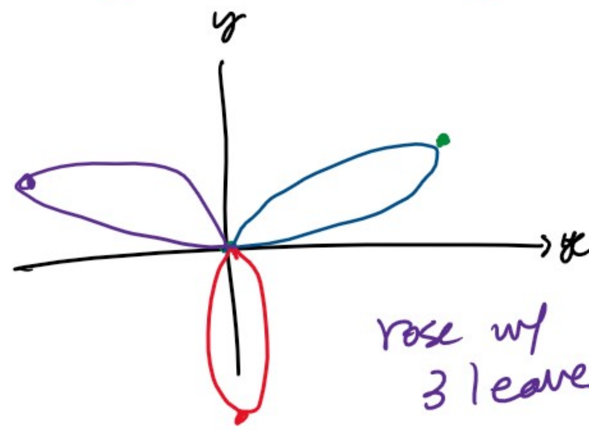


$\boxed{(c)}$



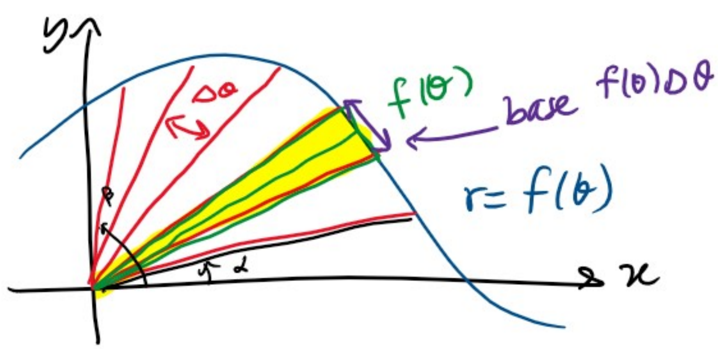
$r = 4\sin(3\theta)$





rose w/ 3 leaves

I. Areas of Regions bounded by Polar Curves



Idea: slice + sum

slice up in θ

area of wedge $\approx$ triangle

$= \frac{1}{2}(\text{base}) \times (\text{height})$

$= \frac{1}{2}(f(\theta)\Delta\theta)(f(\theta)) = \frac{1}{2}[f(\theta)]^2 \Delta\theta$

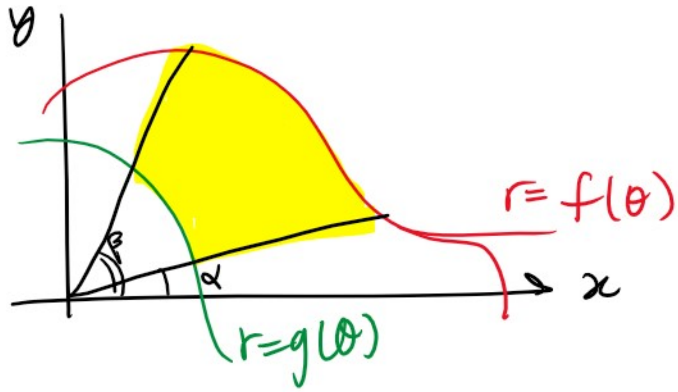
$= \frac{1}{2}(f(\theta)\Delta\theta)(f(\theta)) = \frac{1}{2}[f(\theta)]^2 \Delta\theta$

$\text{Area} \approx \sum \frac{1}{2}[f(\theta)]^2 \Delta\theta \xrightarrow[\Delta\theta \to 0]{\lim}$

$$\boxed{\text{Area} = \int_\alpha^\beta \frac{1}{2}[f(\theta)]^2 \, d\theta}$$

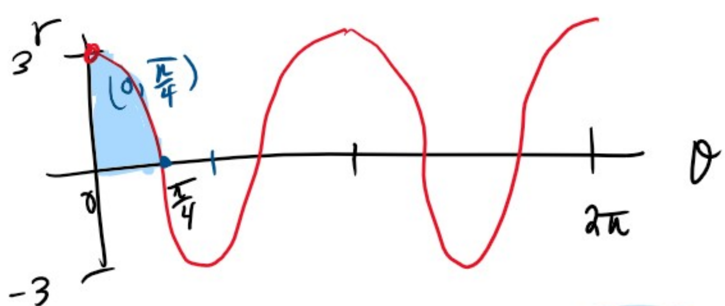
<u>Two Curves:</u>

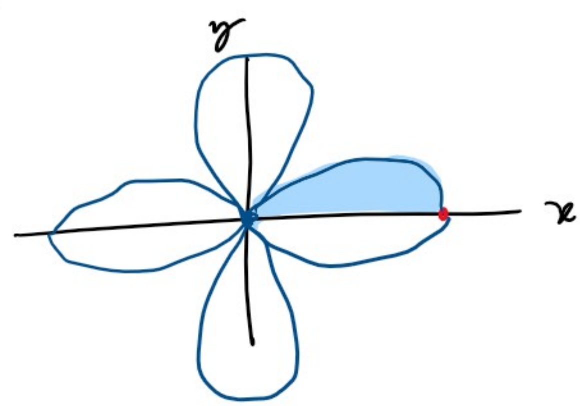
$$\text{Area} = \int_\alpha^\beta \frac{1}{2}\left[f(\theta)^2 - g(\theta)^2\right] d\theta$$



<u>Ex (1)</u>: Find the area of the four-leaf rose

$r = f(\theta) = 3\cos(2\theta)$





Area = 8 x 

$= 8 \times \int_0^{\pi/4} \frac{1}{2}[f(\theta)]^2 \, d\theta$

$= 8 \int^{\pi/4} \frac{1}{2}[3\cos(2\theta)]^2 \, d\theta$

Half Angle

$$= 8\int_0^{\frac{\pi}{4}} \frac{1}{2}\left[3\cos(2\theta)\right]^2 d\theta$$

$$= 4\cdot 9\int_0^{\frac{\pi}{4}} \cos^2(2\theta)\,d\theta$$

Half Angle

$$\cos^2(\varphi) = \frac{1+\cos(2\varphi)}{2}$$

$$\varphi = 2\theta$$

$$2\varphi = 4\theta$$

$$= 36\int_0^{\frac{\pi}{4}} \frac{1+\cos(4\theta)}{2}$$

$$= 18\int_0^{\frac{\pi}{4}} 1+\cos(4\theta)\,d\theta$$

$$= 18\left[\theta + \frac{\sin(4\theta)}{4}\right]_0^{\frac{\pi}{4}}$$

$$= 18\left[\frac{\pi}{4} + \cancel{\frac{\sin(4\frac{\pi}{4})}{4}}^{\,0} - 0 - \cancel{\frac{\sin(0)}{4}}^{\,0}\right]$$

$$= \frac{18\pi}{4} = \boxed{\frac{9\pi}{2}}$$

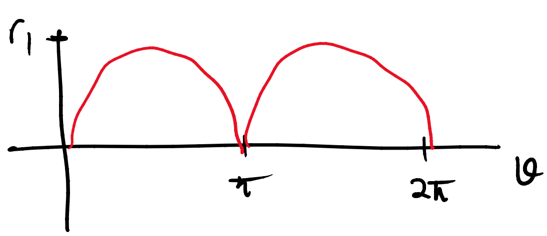
HOTSEAT: Find the area inside the curve $r = \sin^2\theta$

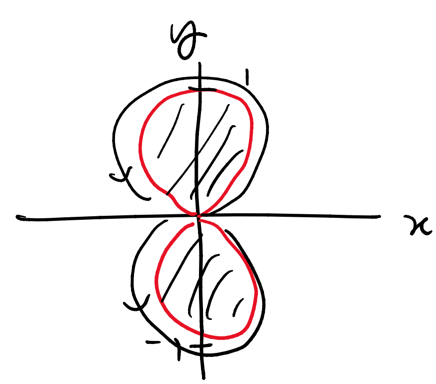
(a) $\frac{\pi}{2}$

(b) $\boxed{\frac{3\pi}{8}}$

(c) 2π

(d) $\frac{3\pi}{4}$





$$\frac{1}{2}\int_0^{2\pi}\left[\sin^2\theta\right]^2 d\theta$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$A = \frac{1}{2}\int_0^{2\pi} \left[\sin^2\theta\right]^2 d\theta \qquad \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$= \frac{1}{2}\int_0^{2\pi} \left[\frac{1-\cos(2\theta)}{2}\right]^2 d\theta$$

$$= \frac{1}{2\cdot 4}\int_0^{2\pi} 1 - 2\cos(2\theta) + \cos^2(2\theta)\, d\theta$$

$$\cos^2\varphi = \frac{1+\cos(2\varphi)}{2}$$

$$\varphi = 2\theta \qquad 2\varphi = 4\theta$$

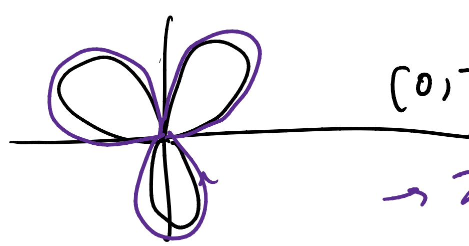
$$= \frac{1}{8}\int_0^{2\pi} 1 - 2\cos(2\theta) + \left[\frac{1+\cos(4\theta)}{2}\right] d\theta$$

$$= \frac{1}{8}\int_0^{2\pi} \frac{3}{2} - 2\cos(2\theta) + \frac{\cos(4\theta)}{2}\, d\theta$$

$$= \frac{1}{8}\left[\frac{3\theta}{2} - 2\,\frac{\sin(2\theta)}{2} + \frac{\sin(4\theta)}{2\cdot 4}\right]_0^{2\pi}$$

$$= \frac{1}{8}\left[\frac{3}{\cancel{2}}(\cancel{2}\pi) - 2\,\cancel{\frac{\sin(2\cdot 2\pi)}{2}}^{0} + \cancel{\frac{\sin(4\cdot 2\pi)}{8}}^{0} - 0 + 2\,\cancel{\frac{\sin(0)}{2}}^{0} - \cancel{\frac{\sin(0)}{8}}^{0}\right]$$

$$= \boxed{\frac{3\pi}{8}}$$



$[0, \pi)$

$\rightarrow$ Integra $\frac{1}{2}\int_0^{\pi} f(\theta)^2\, d\theta$

II Arc length :

II. Arc Length:

The arc length L of the polar curve $r = f(\theta)$ on $[\alpha, \beta]$ is

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

Ex(2): Find the arc length of the cardioid $r = 1 + \cos\theta$ on $0 \le \theta \le 2\pi$

$f(\theta) = 1 + \cos\theta$ $\qquad$ $f'(\theta) = -\sin\theta$

$$L = \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{1 + 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{=1}}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\, d\theta$$

Half angle

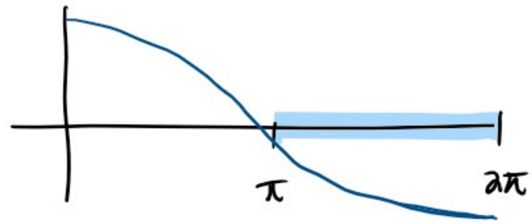
$$\cos^2\varphi = \frac{1+\cos(2\varphi)}{2}$$

$$\boxed{4\cos^2\varphi} = 2 + 2\cos(2\varphi)$$

let $2\varphi = \theta$

$\varphi = \frac{\theta}{2}$

$$= \int_0^{2\pi} \left| 2\cos\left(\frac{\theta}{2}\right) \right| d\theta$$



$$= \int_0^{\pi} 2\cos\left(\frac{\theta}{2}\right) d\theta + \int_{\pi}^{2\pi} \left(-2\cos\left(\frac{\theta}{2}\right)\right) d\theta$$

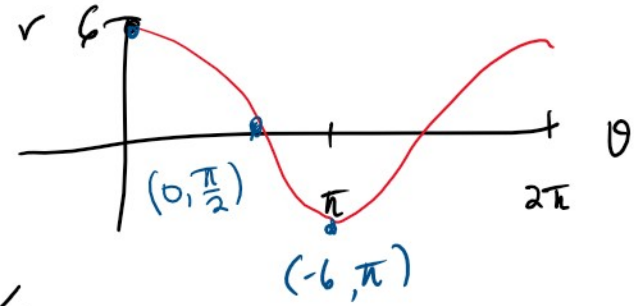
$$= \left[\frac{2\sin\left(\frac{\theta}{2}\right)}{1/2} \right]_0^{\pi} + \left[-\frac{2\sin\left(\frac{\theta}{2}\right)}{1/2} \right]_{\pi}^{2\pi}$$

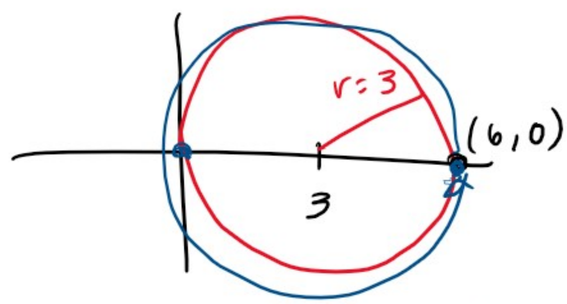
$$= 4\sin\left(\frac{\theta}{2}\right) \Big|_0^{\pi} + \left[-4\sin\left(\frac{\theta}{2}\right) \right]_{\pi}^{2\pi}$$

$$= 4\cancelto{1}{\sin\left(\frac{\pi}{2}\right)} - 4\cancelto{0}{\sin(0)} - 4\cancelto{0}{\sin\left(\frac{2\pi}{2}\right)} + 4\cancelto{1}{\sin\left(\frac{\pi}{2}\right)}$$

$$= 4 + 4 = \boxed{8}$$

Find the arc length of the circle $r = 6\cos\theta$



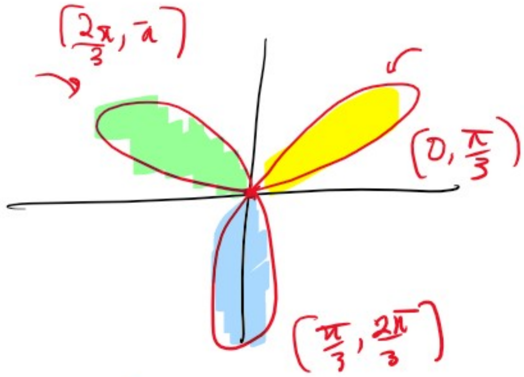


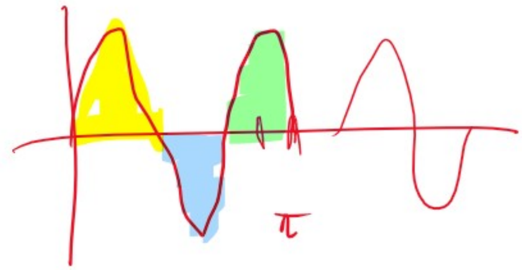
$2\pi r$

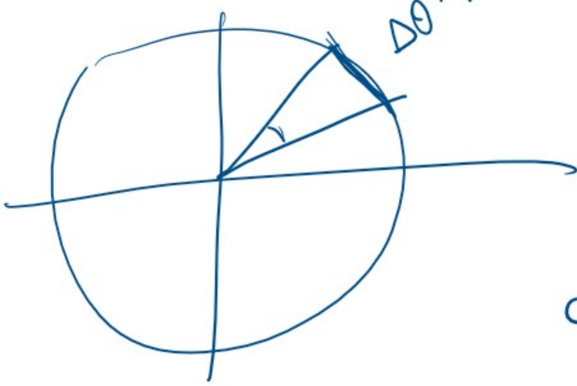
$\hookrightarrow L = 6\pi$

1 revolution in $[0, \pi]$

$$L = \int_0^{\pi} \sqrt{f(\theta)^2 + [f'(\theta)]^2} \, d\theta$$







circunference $2\pi \cdot r$

$\Delta\theta \cdot r$