★ 13.3 Dot Products

GOALS:

- compute dot products
- find angles between vectors
- orthogonal projections
- calculate work done

Announcements:

Monday Jan 17 is a holiday
  - no class
  - no office Hours

HW 1 & 2 due Wed Jan 19

Quiz 1 due Thurs Jan 20

Warm Up:

What is the normal vector $\vec{n}$ of the plane

$$2(x-3) + 3(y-4) + 4(z-5) = 0 \qquad P_0 = (3,4,5)$$

$\boxed{★ \text{(a)} \quad \vec{n} = \langle 2, 3, 4 \rangle}$ (c) $\vec{n} = \langle -2,-3,-4 \rangle$

(b) $\vec{n} = \langle 3, 4, 5 \rangle$ (d) $\vec{n} = \langle -3,-4,-5 \rangle$

I. Dot Product

Def: Given two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, the dot product is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \leftarrow \text{scalar}$$

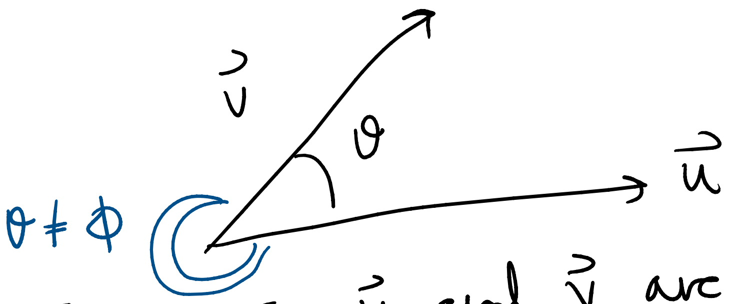
NOTE: the dot product of 2 vectors is a scalar

Ex: Let $\vec{u} = \langle 4, 1, -2 \rangle$
$\vec{v} = \langle 3, 4, 5 \rangle$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$
$$= 4 \cdot 3 + 1 \cdot 4 + (-2)5$$
$$= 12 + 4 - 10 = 6$$

$$\boxed{\vec{u} \cdot \vec{v} = 6}$$

II Geometric Meaning



Thm: If $\vec{u}$ and $\vec{v}$ are nonzero vectors, then

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}| \cos\theta$ where $0 \le \theta \le \pi$

We can use this to calculate the angle $^{\theta}$ between between $\vec{u}$ and $\vec{v}$

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}| \cos\theta$$

$$\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \cos\theta$$

$$\cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \theta$$

Ex: Let $\vec{u} = \langle\sqrt{3}, 1, 0\rangle$ Find the angle θ between $\vec{u}$ and $\vec{v}$

$\vec{v} = \langle 1, \sqrt{3}, 0\rangle$

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

$$\vec{u}\cdot\vec{v} = (\sqrt{3})(1) + (1)(\sqrt{3}) + 0\cdot 0 = 2\sqrt{3}$$

$$|\vec{u}| = \sqrt{(\sqrt{3})^2 + (1)^2 + 0^2} = \sqrt{4} = 2$$

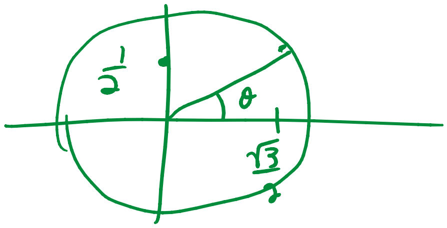
$$|\vec{v}| = \sqrt{1^2 + (\sqrt{3})^2 + 0^2} = \sqrt{4} = 2$$

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \cos^{-1}\left(\frac{2\sqrt{3}}{2\cdot 2}\right)$$

$$= \cos^{-1}\left(\frac{\sqrt{3}}{}\right) = \boxed{\frac{\pi}{} = \theta}$$

$$= \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{\pi}{6} = \theta}$$

unit circle



POLL: If $\vec{u}\cdot\vec{v} = 0$, what is θ?

(a) $\theta = 0$

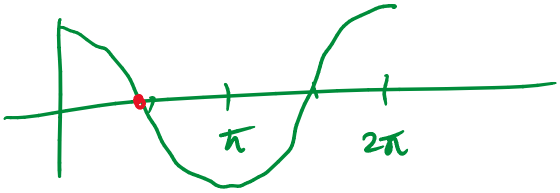
(b) $\theta = \frac{\pi}{4}$

(c) $\theta = \frac{\pi}{2}$

(d) $\theta = \pi$

$$\vec{u}\cdot\vec{v} = 0 = |\vec{u}||\vec{v}|\cos\theta$$

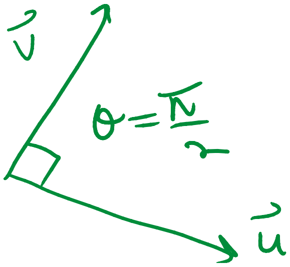
$$\Rightarrow \quad \cos\theta = 0$$



$\cos(\frac{\pi}{2}) = 0$

$$\boxed{\theta = \frac{\pi}{2}}$$

Geometrically



$\vec{u}$ and $\vec{v}$ are at a right angle (perpendicular)

Def: Two vectors are orthogonal if and only if $\vec{u}\cdot\vec{v} = 0$

Notation: $\vec{u} \perp \vec{v}$

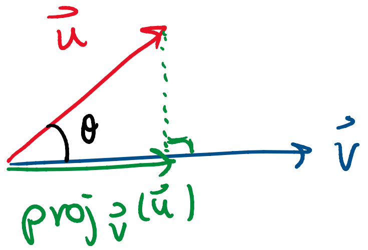
NOTE: in 2D and 3D, two nonzero, orthogonal vectors are perpendicular to each other.

III. Orthogonal Projections:

Q: Given vectors $\vec{u}$ and $\vec{v}$, how closely aligned

Q: Given vectors $\vec{u}$ and v, how closely
are they?

How much of $\vec{u}$ points in the direction of $\vec{v}$



$\text{proj}_{\vec{v}}(\vec{u})$ is how much of $\vec{u}$ is in the $\vec{v}$ direction

Def: The orthogonal projection of $\vec{u}$ onto $\vec{v}$ denoted $\text{proj}_{\vec{v}}(\vec{u})$, where $\vec{v} \neq \vec{0}$, is

$$\text{proj}_{\vec{v}}(\vec{u}) = \underbrace{|\vec{u}|\cos\theta}_{\substack{\text{scalar} \\ \text{"length"}}} \underbrace{\left(\frac{\vec{v}}{|\vec{v}|}\right)}_{\substack{\text{unit vector} \\ \text{"direction"}}}$$

Can also use the formula

$$\text{proj}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

$$= \left(\frac{|\vec{u}|\cancel{|\vec{v}|}\cos\theta}{\cancel{|\vec{v}|}|\vec{v}|\,\cancel{\cos(0)}^{1}}\right)\vec{v} = |\vec{u}|\cos\theta\left(\frac{\vec{v}}{|\vec{v}|}\right)$$

* this formula is nice b/c we don't need to calculate θ

Ex: Let $\vec{u} = \langle 4, 1\rangle$ Find $\text{proj}_{\vec{v}}(\vec{u})$
$\vec{v} = \langle 3, 4\rangle$

$$\text{proj}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

$$= \frac{(4\cdot 3 + 1\cdot 4)}{(3\cdot 3 + 4\cdot 4)} \langle 3,4\rangle$$

$$= \left(\frac{12+4}{9+16}\right)\langle 3,4\rangle = \boxed{\frac{16}{25}\langle 3,4\rangle}$$

IV. Applications of the Dot Product

Def: Let a constant force $\vec{F}$ be applied to an object, producing a displacement $\vec{d}$. If the angle between $\vec{F}$ and $\vec{d}$ is θ, then the work done by the force is

$$W = |\vec{F}||\vec{d}|\cos\theta = \vec{F}\cdot\vec{d}$$

EX: A force $\vec{F} = \langle 3,3,2\rangle$ (in Newtons) moves an object along a line segment from $P = (1,1,0)$ to $Q = (6,6,0)$ (in meters). What is the work done by the force?

$$W = \vec{F}\cdot\vec{d}$$

Need to find $\vec{d} = \overrightarrow{PQ} = \langle 6-1, 6-1, 0-0\rangle = \langle 5,5,0\rangle$

$$W = \vec{F}\cdot\vec{d} = \langle 3,3,2\rangle \cdot \langle 5,5,0\rangle$$

$$= 3\cdot 5 + 3\cdot 5 + 2\cdot 0 = 15+15+0$$

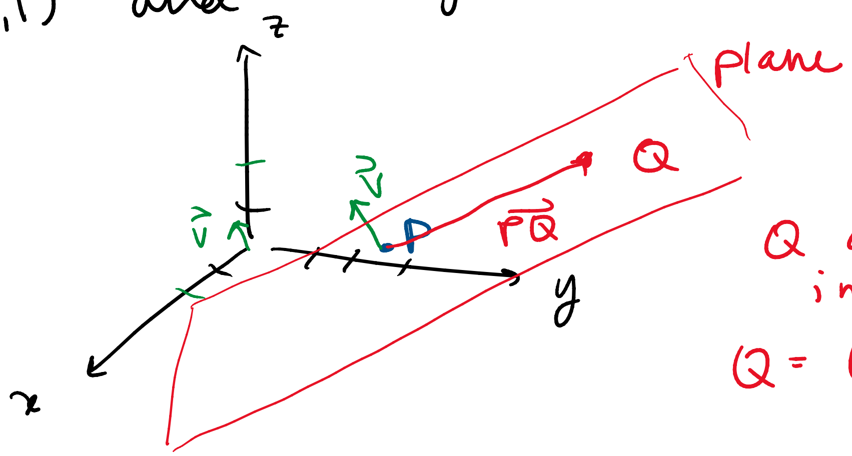
$$\boxed{W = 30 \text{ J}}$$

Another application

EX: Find the equation of the plane passing through orthogonal to $\vec{v} = \langle 2,1,2\rangle$

P $(1,3,1)$ and

Ex: Find the equa
$P = (1,3,1)$ and orthogonal to $\vec{v} = \langle 2,1,2 \rangle$



Q a point in the plane

$(Q = (x,y,z))$

want $\vec{PQ} \perp \vec{v}$

$\langle x-1, y-3, z-1 \rangle \perp \langle 2, 1, 2 \rangle$

so $\vec{PQ} \cdot \vec{v} = 0$

$$(x-1)(2) + (y-3)(1) + (z-1)(2) = 0$$

$$2x - 2 + y - 3 + 2z - 2 = 0$$

$$2x + y + 2z = 2 + 3 + 2 = 7$$

$$\boxed{2x + y + 2z = 7}$$

★ Summary:

- dot product

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$
$$= |\vec{u}|\,|\vec{v}| \cos\theta$$

- to calculate angle θ

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}\right)$$

$\Longrightarrow$ $\vec{u} \cdot \vec{v} = 0$

- orthogonal vectors $\iff$ $\vec{u} \cdot \vec{v} = 0$
- $\text{proj}_{\vec{v}}(\vec{u}) = \left(\dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v}$
- Work $W = |\vec{F}||\vec{d}| \cos\theta = \underset{\text{force}}{\vec{F}} \cdot \underset{\text{displacement.}}{\vec{d}}$