13.4 Cross Products

Announcements:

- HW 1 + 2 due today by 11:59pm
- Quiz 1 due tomorrow

GOALS:

- compute cross products
- find areas of parallelograms + triangles
- identify collinear points
- find orthogonal vectors
- Solve applications

$\vec{u} \perp \vec{v}$ if $\vec{u} \cdot \vec{v} = 0$

Warm Up: Consider the vector $\vec{v} = \langle 1, 2, 3 \rangle$

Which of the following vectors is orthogonal to $\vec{v}$?

(a) $\vec{u} = \langle 2, -1, 0 \rangle$

$\vec{u} \cdot \vec{v} = 1 \cdot 2 + (-1) \cdot 2 + 0 \cdot 3 = 0$

(b) $\vec{w} = \langle 0, 3, 2 \rangle$

I. Cross Product:

Def: the determinant of a 2×2 matrix

$A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is $|A| = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

$$= a_1 b_2 - a_2 b_1$$

The determinant of a 3×3 matrix is:

expansion by minors

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

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$\vec{i} = \langle 1, 0, 0 \rangle$ — unit vector in x-direction

$\vec{j} = \langle 0, 1, 0 \rangle$ — " y-dir

$\vec{k} = \langle 0, 0, 1 \rangle$ — " z-dir

Ex: we can write any vector in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$

$$\vec{v} = \langle 3, 7, 2 \rangle = 3\langle 1, 0, 0 \rangle + 7\langle 0, 1, 0 \rangle + 2\langle 0, 0, 1 \rangle$$

$$= 3\vec{i} + 7\vec{j} + 2\vec{k}$$

Def: The cross-product of 2 vectors

$\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$ = expansion by minors

$$= \vec{i} \underbrace{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}} - \vec{j} \underbrace{\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}}_{\text{scalars}} + \vec{k} \underbrace{\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}}$$

vectors

= vector

Ex: $\vec{u} = \langle -1, 0, 6 \rangle$ $\quad \vec{v} = \langle 2, -5, -3 \rangle$

$\vec{u} \times \vec{v} = | \; \hat{i} \quad \hat{j} \quad \hat{k} \; |$

Ex

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 6 \\ 2 & -5 & -3 \end{vmatrix}$$

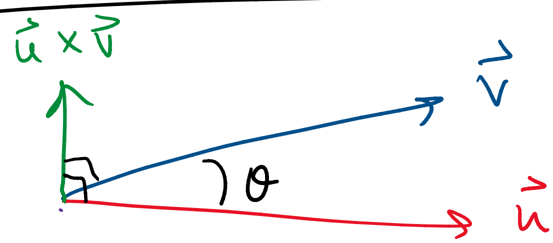
$$= \hat{i}\begin{vmatrix} 0 & 6 \\ -5 & -3 \end{vmatrix} - \hat{j}\begin{vmatrix} -1 & 6 \\ 2 & -3 \end{vmatrix} + \hat{k}\begin{vmatrix} -1 & 0 \\ 2 & -5 \end{vmatrix}$$

$$= \hat{i}\left(\cancel{0(-3)} - 6(-5)\right] - \hat{j}\left[(-1)(-3) - 6(2)\right] + \hat{k}\left[(-1)(-5) - \cancel{0\cdot 2}\right]$$

$$= 30\vec{i} - \vec{j}\,[-9] + \hat{k}\,[5]$$

$$= 30\vec{i} + 9\vec{j} + 5\hat{k} = \boxed{\langle 30, 9, 5\rangle = \vec{u}\times\vec{v}}$$

II. Geometric Interpretation:



$\vec{u} \perp \vec{u}\times\vec{v}$

$\vec{v} \perp \vec{u}\times\vec{v}$

Right Hand Rule

magnitude

$$|\vec{u}\times\vec{v}| = |\vec{u}|\,|\vec{v}|\sin\theta$$

Thm: (Geometry)

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors

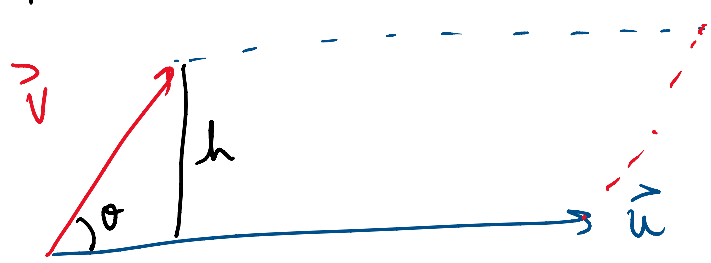
1. $\vec{u} \parallel \vec{v}$ if and only if $\vec{u}\times\vec{v} = \vec{0}$

   $(\sin(0) = \sin(\pi) = 0)$

2. If $\vec{u}$ and $\vec{v}$ are 2 sides of a parallelogram, then the area is

parallelogram, then the area is

$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$



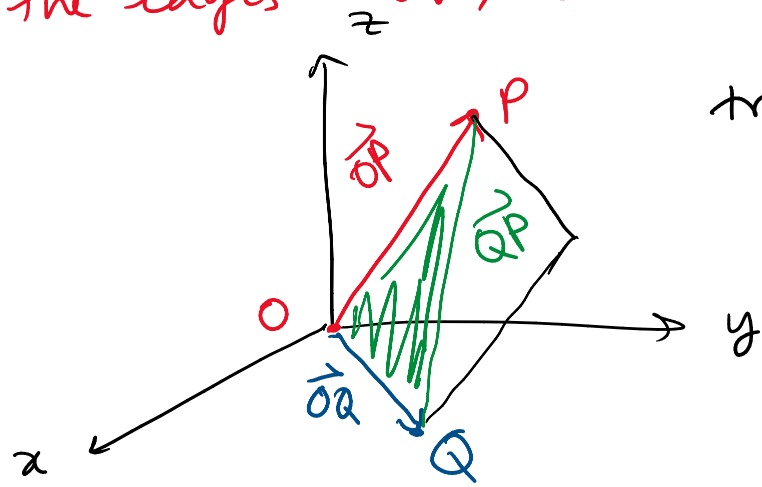
$\sin\theta = \dfrac{h}{|\vec{v}|}$

$h = |\vec{v}| \sin\theta$

Area = base $\times$ height $= |\vec{u}|\, h$

$= |\vec{u}||\vec{v}| \sin\theta = |\vec{u} \times \vec{v}|$

POLL: Consider a triangle defined by the edges $\vec{OP}$, $\vec{OQ}$, and $\vec{PQ}$



triangle $= \frac{1}{2}$ parallelogram

(a) $|\vec{OP} \times \vec{OQ}|$

(b) $\boxed{\frac{1}{2} |\vec{OP} \times \vec{OQ}|}$

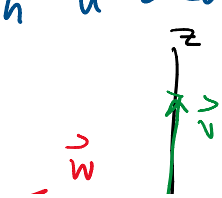
(c) $2|\vec{OP} \times \vec{OQ}|$

NOTE: We can use the cross product to find a vector orthogonal to 2 vectors

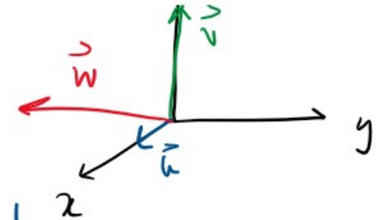
Ex: $\vec{u} = \vec{i}$ and $\vec{v} = 3\vec{k}$

Find $\vec{w}$ that $\perp$ to both $\vec{u}$ and $\vec{v}$

$\vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \end{vmatrix}$



$$\vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$



$$= \hat{i} \begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

$$= \hat{i}\,\cancelto{0}{[0 \cdot 3 - 0 \cdot 0]} - \vec{j}\,(1 \cdot 3 - 0 \cdot 0) + \vec{k}\,\cancelto{0}{(1 \cdot 0 - 0 \cdot 0)}$$

$$= -\vec{j}\,(3) = -3\vec{j} = \langle 0, -3, 0 \rangle$$

We can apply this principle to find equations of planes

$\underline{Ex}$: Find the equation of the plane parallel to $\vec{u} = \vec{i}$ and $\vec{v} = 3\vec{k}$ that goes through point $P = (5, 7, 1)$

Recall. eq of a plane $\vec{n} = \langle a, b, c \rangle$
$P = (5, 7, 1)$

$$a(x-5) + b(y-7) + c(z-1) = 0$$

Find normal vector $\vec{n} = \langle a, b, c \rangle$

plane is parallel to $\vec{u}$, and $\vec{v}$
want: $\vec{n} \perp \vec{u}$ and $\vec{n} \perp \vec{v}$

$\vec{n} = \vec{u} \times \vec{v} = -3\vec{j}$ (last example)

$a = 0$, $b = -3$, $c = 0$

$$\cancelto{0}{0(x-5)} - 3(y-7) + \cancelto{0}{0(z-1)} = 0$$

$0(\cancel{x-5}) - 3(y-7) + 0(\cancel{z-1}) = 0$

$(5, 7, 1)$

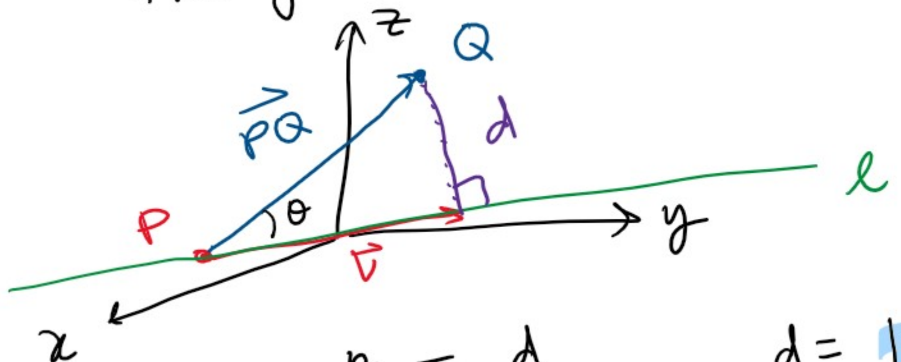
parallel to xz-plane $\rightarrow$ $\boxed{y = 7}$

- distance between a point and a line

point Q

line ℓ through point P

$\vec{r} = \vec{r}_0 + t\vec{v}$ line is parallel to $\vec{v}$



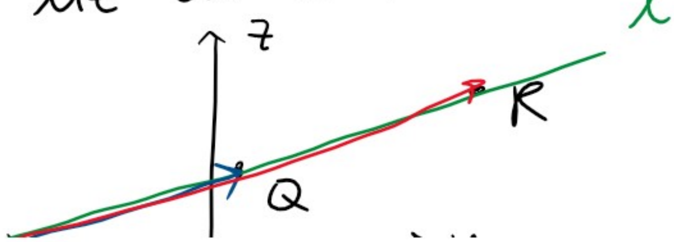
$\sin\theta = \dfrac{d}{|\vec{PQ}|}$ $\qquad d = |\vec{PQ}|\sin\theta$

recall: $|\vec{v} \times \vec{PQ}| = |\vec{v}||\vec{PQ}|\sin\theta$

$= |\vec{v}|\, d$

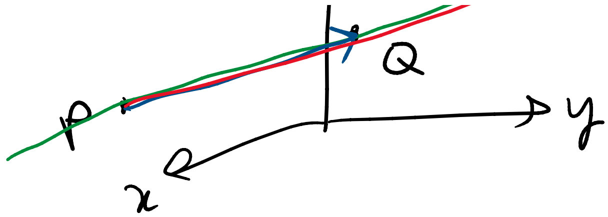
$$\boxed{d = \frac{|\vec{v} \times \vec{PQ}|}{|\vec{v}|}}$$

Def: The points P, Q, and R are collinear if they lie on a line



$\vec{PQ} \parallel \vec{PR}$

$\vec{PQ} \times \vec{PR} = \vec{0}$



$\vec{PQ} \times \vec{PR} = \vec{0}$

POLL: Assume P, Q, R are collinear.
Which is true?

(a) $\boxed{\vec{PQ} \times \vec{PR} = \vec{0}}$

(b) $\vec{PQ} \cdot \vec{PR} = 0$ → $\vec{PQ} \perp \vec{PR}$

(c) $\vec{PQ} \times \vec{PR} = \vec{RQ}$

III. Applications

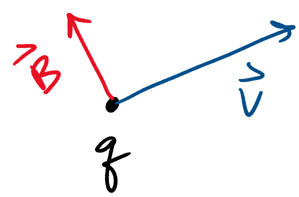
- Torque $\vec{F}$ force applied at head of vector $\vec{r}$

  torque $\vec{\tau} = \vec{r} \times \vec{F}$

- magnetic force on a moving charge

  charge q

  velocity $\vec{v}$



through magnetic field $\vec{B}$

force on the charge $\vec{F} = |q| \, (\vec{v} \times \vec{B})$