Lesson 4 - MA 166 LEC 286

13.4 Cross Products

Announcements:

- HW 1+2 due today by 11:59pm
- Quiz 1 due tomorrow by 11:59pm

GOALS:

- compute cross products
- find areas of parallelograms + triangles
- identify collinear points
- find orthogonal vectors
- Solve applications

$\vec{u} \perp \vec{v}$ if $\vec{u} \cdot \vec{v} = 0$

Warm Up: Consider the vector $\vec{v} = \langle 1,2,3 \rangle$. Which of the following vectors is orthogonal to $\vec{v}$?

(a) $\vec{u} = \langle 2,-1,0 \rangle$

$\vec{u} \cdot \vec{v} = 1 \cdot 2 + 2(-1) + 3 \cdot 0 = 0$

(b) $\vec{w} = \langle 0,3,2 \rangle$

I. Cross Product:

Def: the determinant of a 2x2 matrix is

$$A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \qquad |A| = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \underbrace{a_1 b_2 - a_2 b_1}_{\text{scalar}}$$

The determinant of a 3x3 matrix is

expansion by minors

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 [b_2 c_3 - b_3 c_2] - a_2 [b_1 c_3 - b_3 c_1] + a_3 [b_1 c_2 - b_2 c_1]$$

Def: The coordinate unit vectors are:

Def: The <u>coordinate unit vectors</u> are :

$\vec{i} = \hat{i} = \langle 1,0,0 \rangle$ unit vector in x-direction

$\vec{j} = \hat{j} = \langle 0,1,0 \rangle$ " " y-dir

$\vec{k} = \hat{k} = \langle 0,0,1 \rangle$ " " z-dir

Ex: We can write any vector in terms of $i, \vec{j}, \hat{k}$

$$\vec{V} = \langle 3,7,2 \rangle = 3\langle 1,0,0 \rangle + 7\langle 0,1,0 \rangle + 2\langle 0,0,1 \rangle$$

$$= 3\vec{i} + 7\vec{j} + 2\vec{k}$$

Def: The cross product of 2 vectors

$\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$ expansion by minors

$$= \hat{i} \underbrace{\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}} - \vec{j} \underbrace{\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}}_{\text{scalars}} + \hat{k} \underbrace{\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}}$$

vectors

<u>NOTE</u>: $\vec{u} \times \vec{v}$ = vector

$\vec{u} \cdot \vec{v}$ = scalar

Ex: Let $\vec{u} = \langle -1, 0, 6 \rangle$ $\vec{v} = \langle 2, -5, -3 \rangle$

Ex: Let $\vec{u} = \langle -1, 0, 6 \rangle$ $\quad v = \langle 2, -5, -3 \rangle$

Then $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \vec{j} & \vec{k} \\ -1 & 0 & 6 \\ 2 & -5 & -3 \end{vmatrix}$

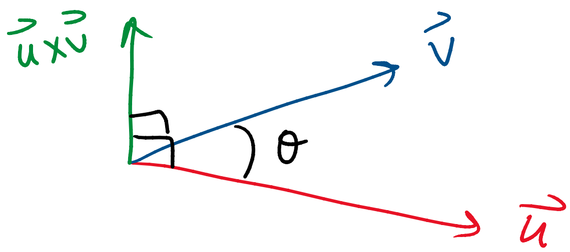
$$= \vec{i} \begin{vmatrix} 0 & 6 \\ -5 & -3 \end{vmatrix} - \vec{j} \begin{vmatrix} -1 & 6 \\ 2 & -3 \end{vmatrix} + \vec{k} \begin{vmatrix} -1 & 0 \\ 2 & -5 \end{vmatrix}$$

$$= \vec{i} \left[\cancelto{0}{0(-3)} - 6(-5) \right] - \vec{j} \left[(-1)(-3) - 6 \cdot 2 \right] + k \left[(-1)(-5) - \cancelto{0}{0 \cdot 2} \right]$$

$$= 30\vec{i} - \vec{j}[-9] + 5\vec{k} = 30\vec{i} + 9\vec{j} + 5\vec{k}$$

$$= \langle 30, 9, 5 \rangle$$

II. Geometric Interpretation:



$\vec{u} \times \vec{v} \perp \vec{u}$

$\vec{u} \times \vec{v} \perp \vec{v}$

Right Hand Rule

- magnitude

$$|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta$$

$$|\vec{u} \cdot \vec{v}| = |\vec{u}|\,|\vec{v}| \cos\theta$$

Thm: (Geometry) Let $\vec{u}$ and $\vec{v}$ be nonzero vectors

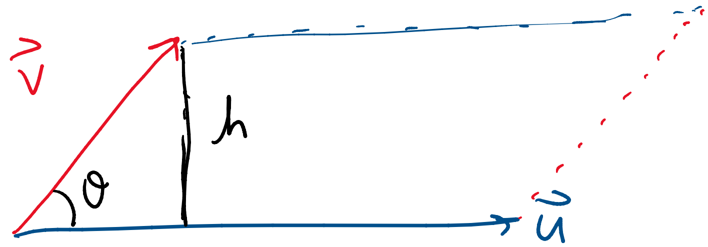
1. $\vec{u} \parallel \vec{v}$ if and only if $\vec{u} \times \vec{v} = \vec{0}$

$$|\vec{u} \times \vec{v}| = |\vec{0}| = 0 = |\vec{u}||\vec{v}| \sin\theta \to \sin\theta = 0$$

$$\theta = 0, \pi$$

2. If $\vec{u}$ and $\vec{v}$ are 2 sides of a parallelogram, then the area is

$$A = |\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$$

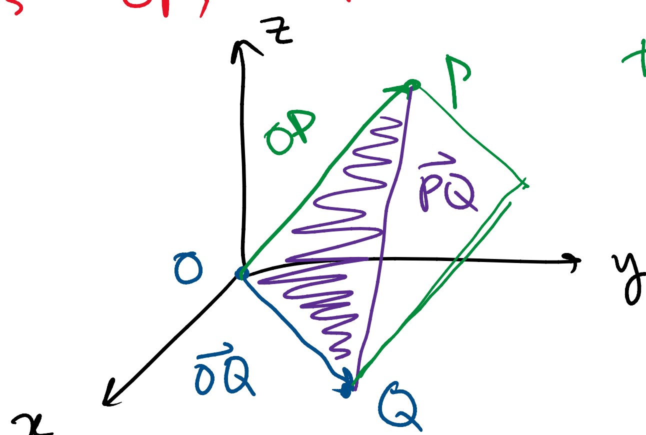


$$\sin\theta = \frac{h}{|\vec{v}|}$$

$$\text{Area} = \text{base} \times \text{height} = |\vec{u}| h$$

$$= |\vec{u}| \underbrace{|\vec{v}| \sin\theta}_{h} = |\vec{u} \times \vec{v}|$$

POLL: Consider a triangle defined by the edges $\vec{OP}$, $\vec{OQ}$ and $\vec{PQ}$



triangle = $\frac{1}{2}$ parallelogram

What is the area of the triangle?

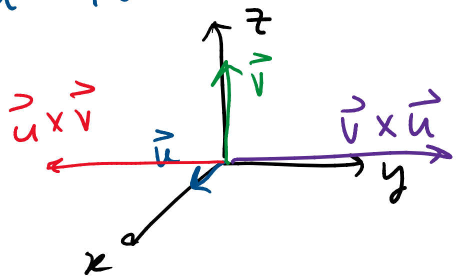
(a) $|\vec{OP} \times \vec{OQ}|$ $\boxed{\text{(b) } \frac{1}{2}|\vec{OP} \times \vec{OQ}|}$ (c) $2|\vec{OP} \times \vec{OQ}|$

We can use the cross product to find a vector orthogonal to 2 vectors

We can use the cross pro[illegible]
vector orthogonal to 2 vectors

Ex: Let $\vec{u} = \vec{i}$ and $\vec{v} = 3\vec{k}$

Find a vector $\vec{w}$ orthogonal to $\vec{u}$ and $\vec{v}$



$$\vec{w} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

(could also use $\vec{w} = \vec{v} \times \vec{u}$)

$$= \vec{i} \begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

$$= \vec{i}\,\cancelto{0}{[0 \cdot 3 - 0 \cdot 0]} - \vec{j}\,[1 \cdot 3 - \cancelto{0}{0 \cdot 0}] + \vec{k}\,[1 \cdot \cancel{0} - \cancel{0 \cdot 0}]$$

$$= -3\vec{j} = \langle 0, -3, 0 \rangle$$

Ex: Find the equation of the plane parallel to $\vec{u} = \vec{i}$ and $\vec{v} = 3\vec{k}$ that goes through the point $P = (5, 7, 1)$

Recall, the eqn of a plane $\vec{n} = \langle a, b, c \rangle$ passing through $P = (5, 7, 1)$

$$a(x-5) + b(y-7) + c(z-1) = 0$$

Need to find $\vec{n} = \langle a, b, c \rangle$

plane $\parallel$ $\vec{u}$ and $\vec{v}$

want $\vec{n} \perp \vec{u}$ and $\vec{n} \perp \vec{v}$

$$\vec{n} = \vec{u} \times \vec{v} = \langle 0, -3, 0 \rangle$$ from last example

$a = 0, \quad b = -3, \quad c = 0$

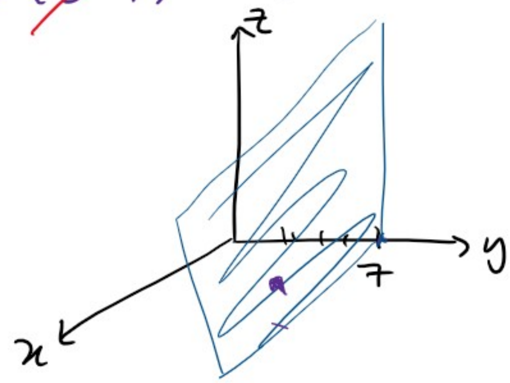
$\vec{n} = u \times v$

$a = 0, \quad b = -3, \quad c = 0$

$$0(\cancel{x-5}) - 3(y-7) + 0(\cancel{z-1}) = 0$$

parallel to x-z plane

$$\boxed{y = 7}$$



$P = (5, 7, 1)$

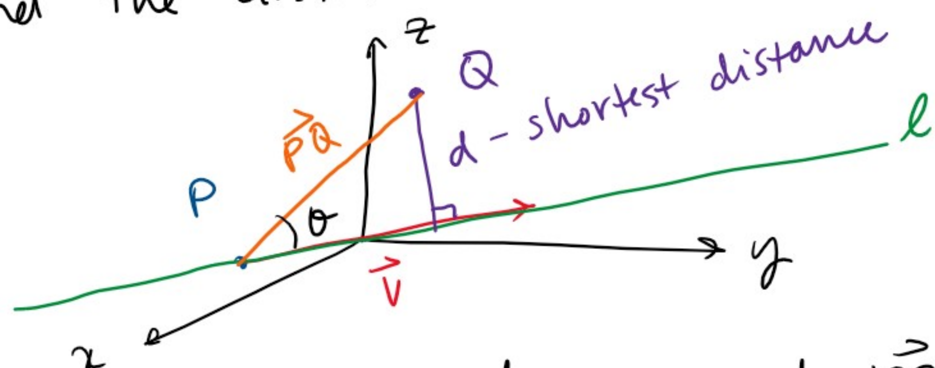
- distance between a point and a line

point Q

line l $\quad \vec{r} = \vec{r}_0 + t\vec{v}$ passes through P

line is parallel to $\vec{v}$

find the distance between Q and l



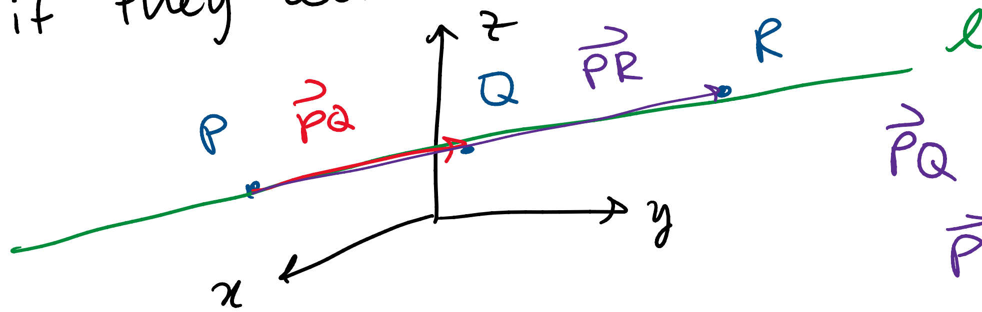
$$\sin\theta = \frac{d}{|\vec{PQ}|} \qquad d = \underbrace{|\vec{PQ}|\sin\theta}$$

$$|\vec{v} \times \vec{PQ}| = |\vec{v}|\underbrace{|\vec{PQ}|\sin\theta}_{d} = |\vec{v}|\, d$$

$$\boxed{d = \frac{|\vec{v} \times \vec{PQ}|}{|\vec{v}|}}$$

collinear

Def: The points P, Q, R are collinear if they all lie on a line



$\vec{PQ} \parallel \vec{PR}$

$\vec{PQ} \times \vec{PR} = \vec{0}$

POLL: Assume, P, Q, R are collinear. Which statements is true

$\vec{PQ} \cdot \vec{PR} = 0$

$\vec{PQ} \perp \vec{PR}$

(a) $\vec{PQ} \times \vec{PR} = \vec{0}$

(b) $\vec{PQ} \cdot \vec{PR} = 0$

(c) $\vec{PQ} \times \vec{PR} = \vec{RQ}$

Applications — read textbook

Torque $\qquad \vec{\tau} = \vec{r} \times \vec{F}$

$\vec{\tau}$ – torque, $\vec{r}$ – length, $\vec{F}$ – force

Magnetic $\qquad |\vec{F}| = |q| \, |\vec{v} \times \vec{B}|$

q - charge

$\vec{v}$ - velocity of q

$\vec{B}$ - mag. field