Announcements:

- please wear your mask correctly during class

☆ 6.3: Volumes by Slicing; Washer Method

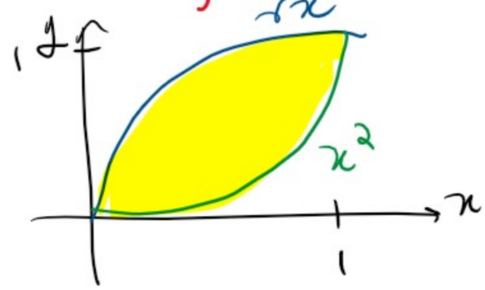
GOALS:

Find volumes of solids using: (1) General Slicing Method
(2) Washer Method

Warm Up: Set up the integral to find the area between the curves $f(x) = \sqrt{x}$ and $g(x) = x^2$

$$\boxed{(a) \quad A = \int_0^1 (\sqrt{x} - x^2)\,dx}$$

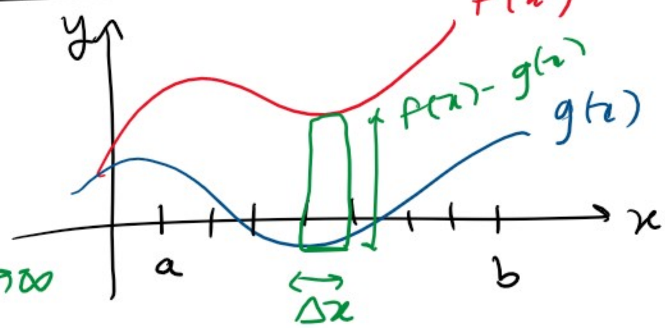
$$(b) \quad A = \int_0^1 (x^2 - \sqrt{x})\,dx$$



Q: How do we find volumes of 3D solids?

I. General Slicing Method:

In 2D



$$\sum_{k=1}^{n} [f(x_k) - g(x_k)]\Delta x$$

lim as $n \to \infty$

$$A = \int_a^b [f(x) - g(x)]\,dx$$

In 3D

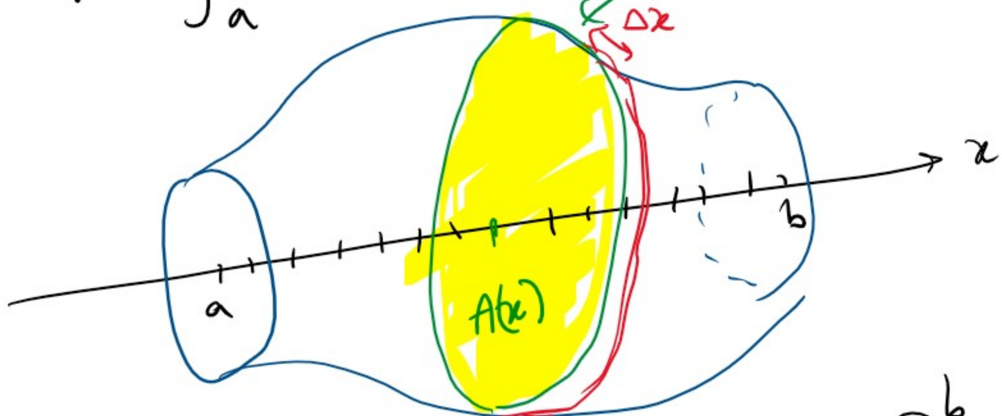
Def: (General Slicing)

... extends from

Def: (General Slicing)
Suppose a solid object extends from $x=a$ to $x=b$, and the cross-section perpendicular to x-axis has area $A(x)$

$$V=\int_a^b A(x)\,dx$$

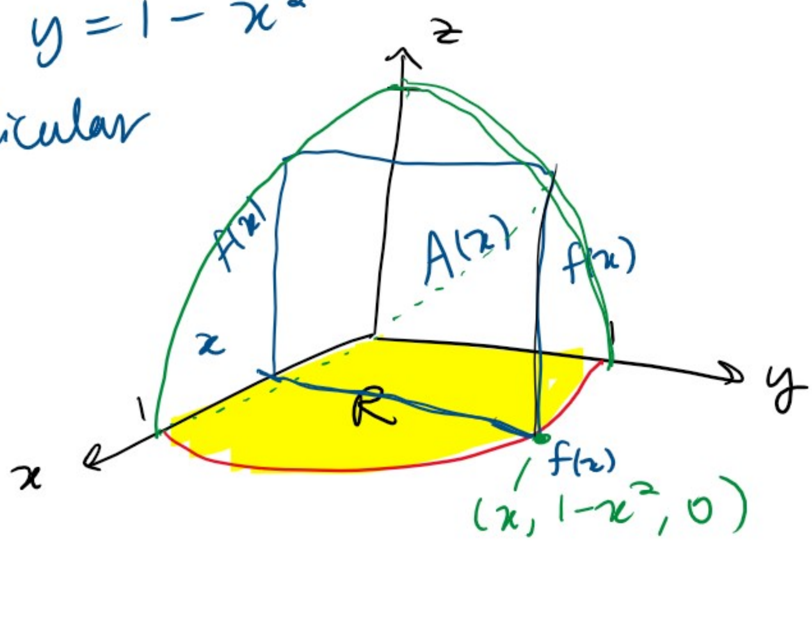
cross-section is an oval



$$V \approx \sum_{k=1}^{n} A(x_k)\,\Delta x \xrightarrow[n\to\infty]{\lim} V=\int_a^b A(x)\,dx$$

Ex: Volume of "parabolic cube"
R - region in 1st quadrant $(x, y \geq 0)$ bounded by x-axis, y-axis and $y=1-x^2$

Solid has perpendicular cross-sections are squares



at x, the area of cross-section $A(x) = f(x) \cdot f(x)$
$= (1-x^2)^2$

so volume of solid

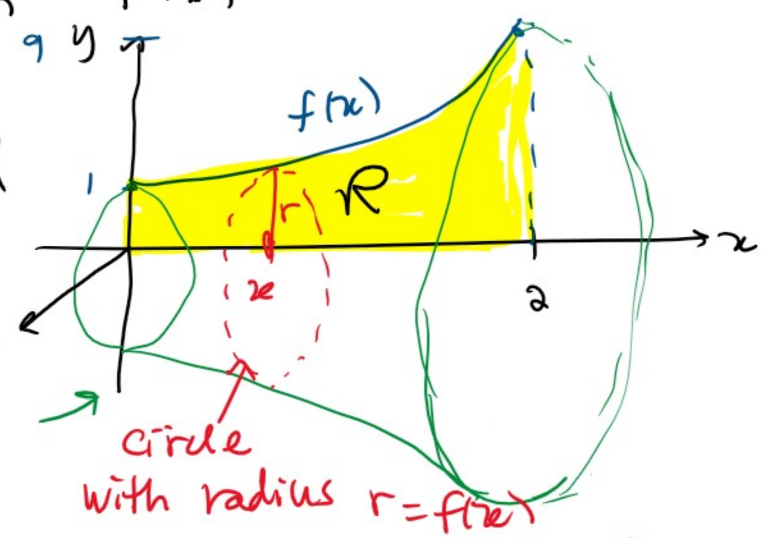
$$V = \int_a^b A(x)\,dx = \int_0^1 (1-x^2)^2\,dx$$
$$= \int_0^1 [1 - 2x^2 + x^4)\,dx$$
$$= \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1$$
$$= 1 - \frac{2}{3} + \frac{1}{5} - 0 = \boxed{\frac{8}{15} = V}$$

II. Disk / Washer Method

Consider a function $f(x) = (x+1)^2$ on $[0,2]$

Take R and relvolve it around the x-axis

solid of revolution



POLL: using the General Slicing Method, What is the area $A(x)$ of the cross-section

(a) $2\pi f(x)$

(c) $\frac{4}{3}\pi [f(x)]^3$

(a) $2\pi f(x)$

$\boxed{\text{(b)} \quad \pi [f(x)]^2}$

(c) $\frac{4}{3}\pi [f(x)]^3$

area of disk πr^2

$r = f(x) \quad A(x) = \pi [f(x)]^2$

So according to the General Slicing Method

$$A(x) = \pi [f(x)]^2$$

$$V = \int_a^b \pi [f(x)]^2 \, dx = \int_0^2 \pi [(x+1)^2]^2 \, dx$$

$$= \int_0^2 \pi (x+1)^4 \, dx$$

use u-substition

let $u = x+1$

$du = \frac{d}{dx}[x+1] = dx$

when $x=0 \quad u=1$

$x=2 \quad u = 2+1 = 3$

$$= \int_1^3 \pi u^4 \, du = \pi \left[\frac{u^5}{5} \right]_1^3$$

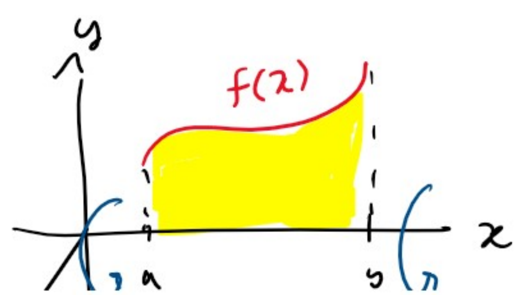
$$= \pi \left[\frac{3^5}{5} - \frac{1^5}{5} \right] = \pi \left[\frac{243-1}{5} \right] = \boxed{\frac{242\pi}{5}}$$

When we have a solid of revolution around the x-axis, then we can use the Disk Method

Disk Method:

$f(x)$ on $[a,b]$

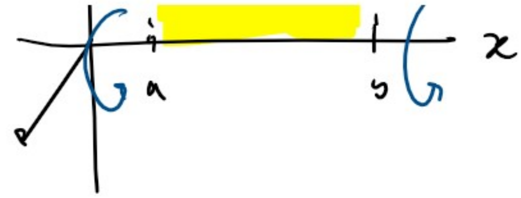
revolved around x-axis



$f(x)$ on $[a,b]$

revolved around x-axis

$$V = \int_a^b \pi \left[f(x)\right]^2 dx$$



III. Washer Method:

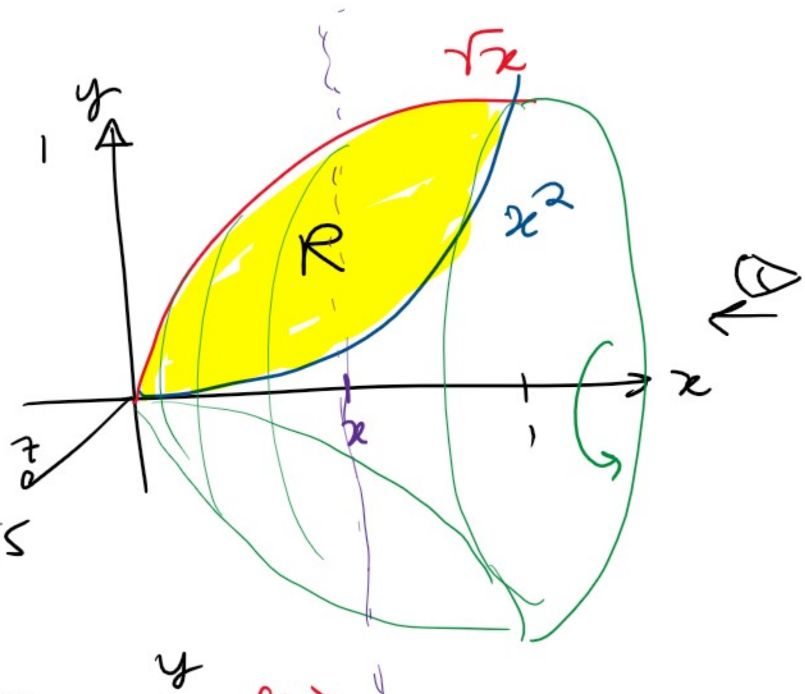
Consider

$f(x) = \sqrt{x}$

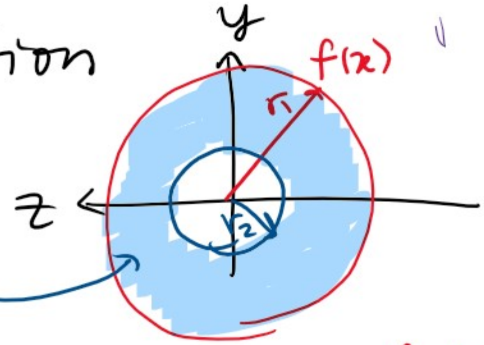
$g(x) = x^2$

on $[0,1]$

revolve around x-axis



Look at a cross-section



$r_2 = g(x)$

A(x) is the area of

POLL: What is the area $A(x)$ of this cross-section?

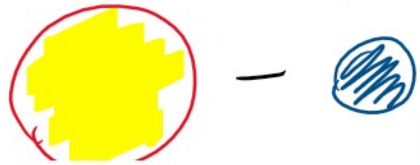
$\boxed{\text{(a)} \quad \pi\left[f(x)\right]^2 - \pi\left[g(x)\right]^2}$

(b) $2\pi f(x) - 2\pi g(x)$

(c) $\pi\left[f(x) - g(x)\right]^2$

$r_1 = f(x)$

$r_2 = g(x)$

$A(x) =$  $= \pi r_1^2 - \pi r_2^2$

$A(x) = \pi\left[f(x)\right]^2 - \pi\left[g(x)\right]^2$

$A(x) =$  $-$

$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2$

$$V = \int_0^1 \left(\pi(\sqrt{x})^2 - \pi(x^2)^2\right] dx$$

$$= \int_0^1 (\pi x - \pi x^4)\,dx = \boxed{\frac{3\pi}{10} = V}$$

<u>Washer Method</u>: $f(x) \geq g(x) \geq 0$ on $[a,b]$

revolve this R around x-axis

$$V = \int_a^b \pi\left[\underbrace{(f(x))^2}_{\text{outer radius}} - \underbrace{(g(x))^2}_{\text{inner radius}}\right] dx$$

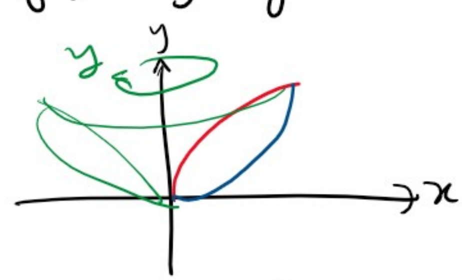
<u>NOTE</u>: Disk method $\longrightarrow$ $g(x) = 0$

<u>NOTE</u>: Revolve functions around y-axis

<u>Washer Method</u>: $p(y) \geq q(y) \geq 0$ on $[c,d]$

revolve around y-axis

$$V = \int_c^d \pi\left[(p(y))^2 - (q(y))^2\right] dy$$



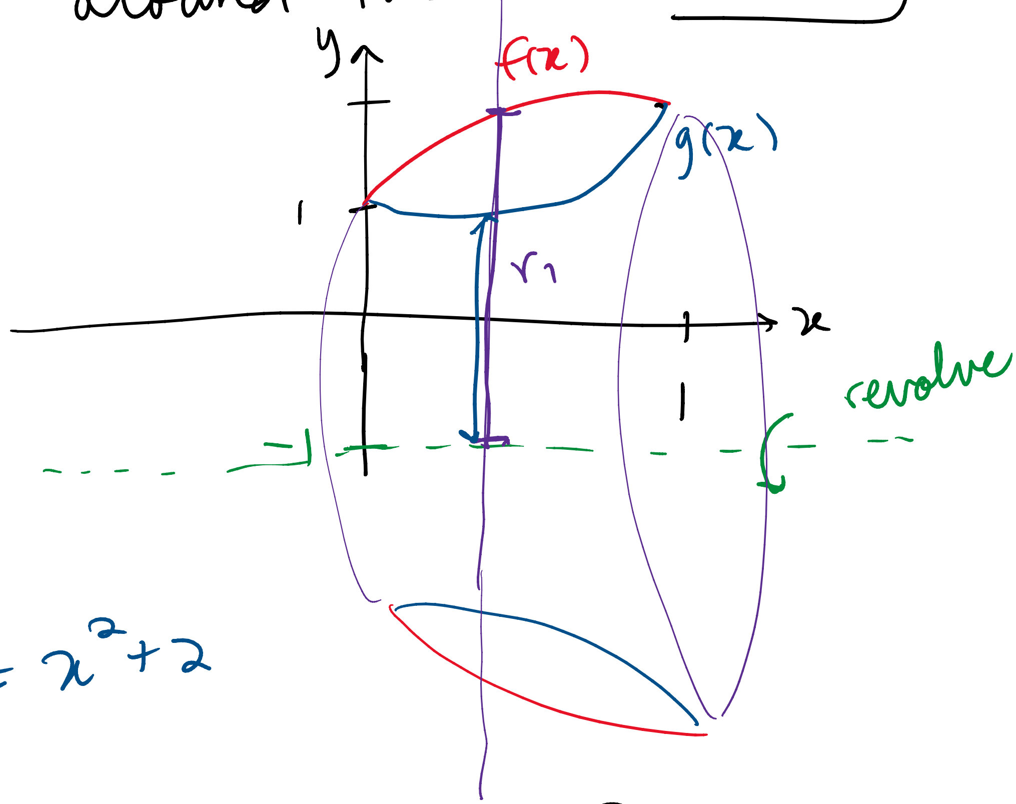
NOTE: We can revolve around lines

NOTE: We can revolve around lines other than the axes

Let $f(x) = \sqrt{x} + 1$ on $[0,1]$

$g(x) = x^2 + 1$

revolve around the line $\boxed{y = -1}$



$A(x) = \pi r_1^2 - \pi r_2^2$

$r_1 = f(x) - (-1)$

$= f(x) + 1$

$= \sqrt{x} + 1 + 1$

$r_1 = \sqrt{x} + 2$

$r_2 = g(x) - (-1)$

$= x^2 + 1 + 1 = x^2 + 2$

$$V = \int_0^1 \left[\pi(\sqrt{x}+2)^2 - \pi(x^2+2)^2\right] dx$$

$$= \pi \int_0^1 \left[x + 4\sqrt{x} - x^4 - 4x^2\right] dx$$

$$= \pi \left[\frac{x^2}{2} + \frac{4x^{3/2}}{3/2} - \frac{x^5}{5} - \frac{4x^3}{3}\right]_0^1$$

$$= \pi \left(\frac{1}{2} + \frac{8}{3} - \frac{1}{5} - \frac{4}{3}\right] = \boxed{\frac{49\pi}{30} = V}$$