☆ 6.3: Volumes by Slicing; Washer Method

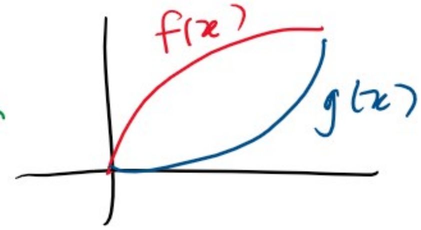
Announcements:
- peer note-taker needed (see announcement on Brightspace)
- please wear your mask correctly during class

GOALS:
Find volumes of solids using: (1) General Slicing Method
(2) Washer Method

Warm Up: Set up the integral to find the area between the curves $f(x) = \sqrt{x}$ and $g(x) = x^2$

(a) $\boxed{A = \int_0^1 (\sqrt{x} - x^2)\,dx}$

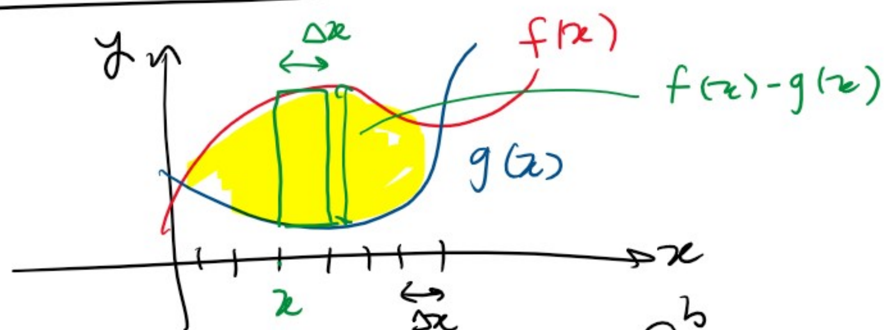
(b) $A = \int_0^1 (x^2 - \sqrt{x})\,dx$



Q: How to find the volume of 3D solids

I. General Slicing Method:

In 2D



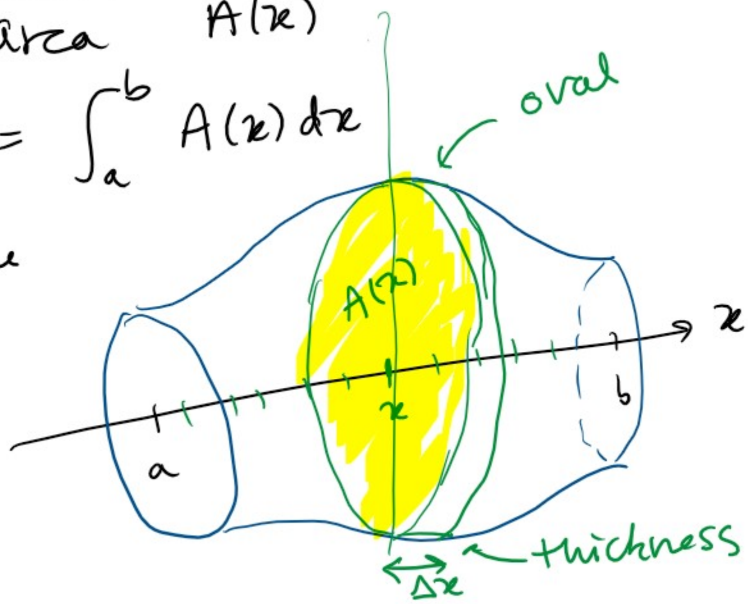
$$A \approx \sum_{k=1}^{n} [f(x) - g(x)]\Delta x \xrightarrow[n\to\infty]{\lim} A = \int_a^b [f(x) - g(x)]\,dx$$

Similar in 3D:

Def: (General Slicing Method)
Suppose a solid object extends from $x=a$ to $x=b$ and the cross-section ... the x-axis has

Suppose a so... and the cross-section
to $x=b$
perpendicular to the x-axis has
area $A(x)$

$$V = \int_a^b A(x)\,dx$$



$A(x)\Delta x$ – volume of slice

$$V \approx \sum_{k=1}^{n} A(x_k)\Delta x$$

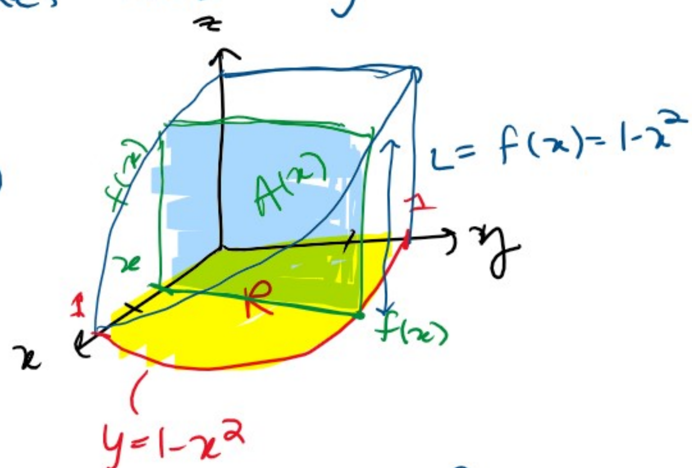
limit $n\to\infty$

$$V = \int_a^b A(x)\,dx$$

Ex: "parabolic curve"

R – region in 1st quadrant $(x\geq 0,\ y\geq 0)$
bounded by axes and $y = 1-x^2$

solid has base R
cross-sections ($\perp$ to x)
that are squares



$$V = \int_a^b A(x)\,dx$$

$$A(x) = \text{base}\cdot\text{height} = L^2 = (1-x^2)^2$$

$$V = \int_0^1 (1-x^2)^2\,dx = \int_0^1 (1-2x^2+x^4)\,dx$$

$$\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1 = 1 - \frac{2}{3} + \frac{1}{5} - 0$$

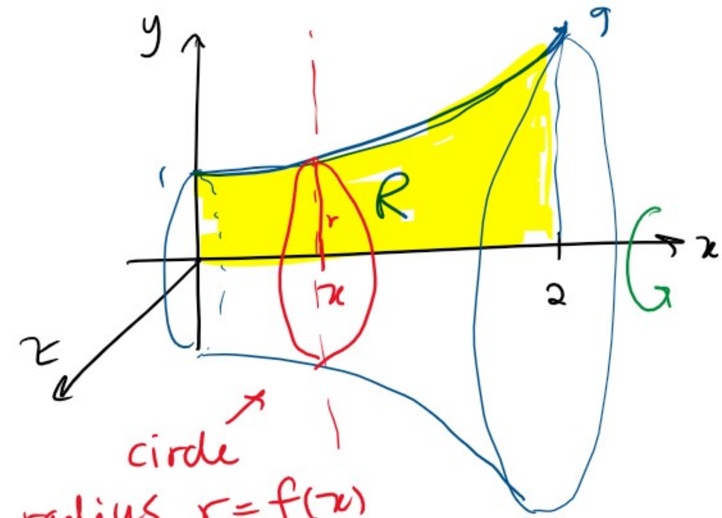
$$= \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1 = 1 - \frac{2}{3} + \frac{1}{5} - 0$$

$$\boxed{V = \frac{8}{15}}$$

II. Disk Method:

function $f(x) = (x+1)^2$ on $[0, 2]$

Take R and revolve around the x-axis



Solid of revolution

POLL: Using the General Slicing Method, what is $A(x)$?

(a) $2\pi f(x)$

(c) $\frac{4}{3}\pi [f(x)]^3$

$$\boxed{\text{(b)} \quad \pi [f(x)]^2}$$

$$A(x) = \pi r^2 = \pi [f(x)]^2$$

$$V = \int_a^b A(x)\,dx = \int_0^2 \pi [f(x)]^2\,dx$$

$$= \int_0^2 \pi [(x+1)^2]^2\,dx = \int_0^2 \pi (x+1)^4\,dx$$

u-substition $\quad u = x+1 \qquad$ when $x = 0 \quad u = 0+1 = 1$

u-substition

let $u = x+1$

$du = dx$

when $x=0$ $\quad u = 0+1 = 1$

$x = 2$ $\quad u = 2+1 = 3$

$$= \int_1^3 \pi u^4 \, du = \pi \left[\frac{u^5}{5} \right]_1^3$$

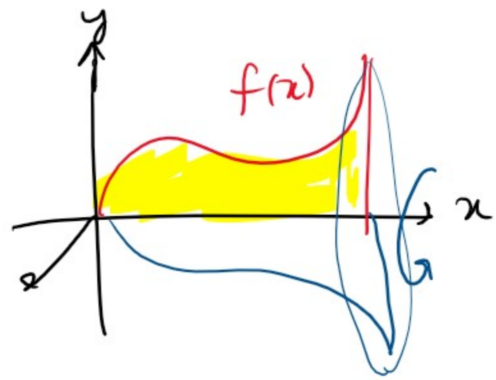
$$= \pi \left[\frac{3^5}{5} - \frac{1^5}{5} \right] = \pi \left[\frac{243-1}{5} \right] = \boxed{\frac{\pi \, 242}{5}}$$

Disk Method:

$f(x)$ on $[a, b]$

revolving around the x-axis

$$V = \int_a^b \pi \left[f(x) \right]^2 dx$$



III. Washer Method:

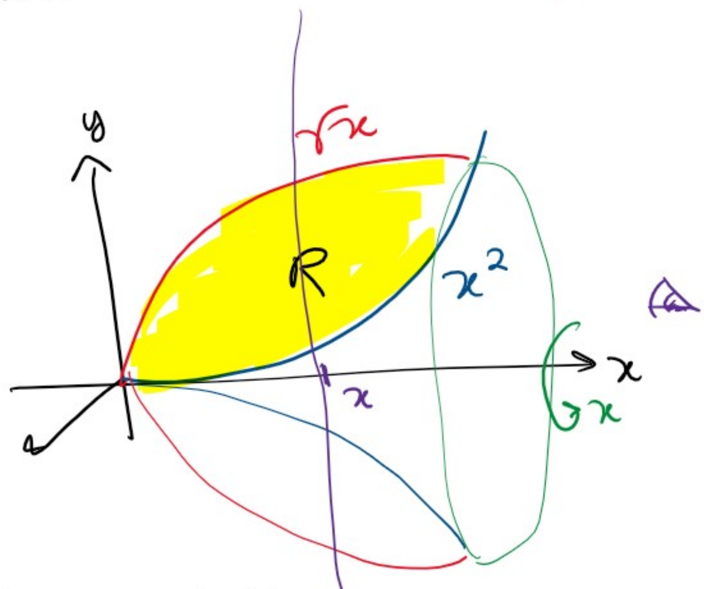
solid of revolution

R – $f(x) = \sqrt{x}$

$g(x) = x^2$

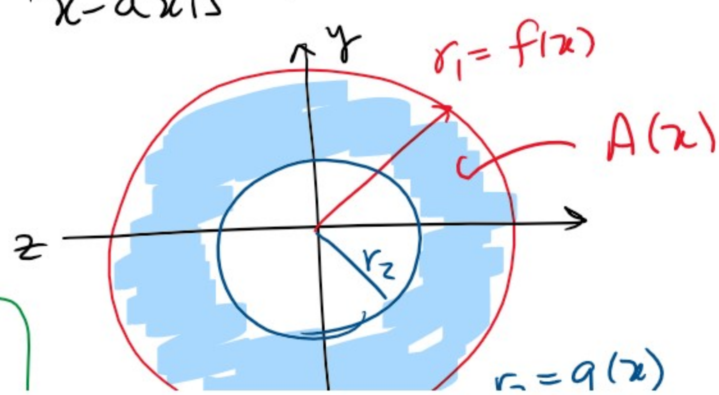
$[0, 1]$

revolve around x-axis



Look at a slice $\perp$ to x-axis

POLL: What is the area $A(x)$ of this cross-section



$A(x) = \pi [f(x)]^2 - \pi [g(x)]^2$

cross-section

$\boxed{(a)\ \pi[f(x)]^2 - \pi[g(x)]^2}$

(b) $2\pi f(x) - 2\pi g(x)$

(c) $\pi[f(x) - g(x)]^2$

$A(x) =$ 

$= \pi r_1^2 - \pi r_2^2$

$= \pi[f(x)]^2 - \pi[g(x)]^2$

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$$

$$= \int_0^1 \pi\left[(\sqrt{x})^2 - (x^2)^2\right]dx$$

$$= \pi\int_0^1 (x - x^4)\,dx = \pi\left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Big|_0^1$$

$$= \pi\left(\frac{5-2}{10}\right) = \boxed{\frac{3\pi}{10}}$$

Washer Method:

$f(x) \geq g(x) \geq 0$ $\quad [a,b]$

revolve region R around x-axis

$$V = \int_a^b \pi\left[\underbrace{(f(x))^2}_{\text{outer radius}} - \underbrace{(g(x))^2}_{\text{inner radius}}\right]dx$$

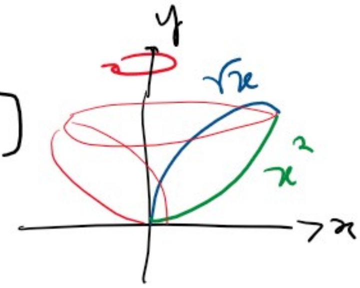
NOTE: Disk Method same ↑ if $g(x) = 0$

NOTE: you can revolve regions around the y-axis

NOTE: you can revolve regions
y-axis

$p(y) \geq q(y) \geq 0$ on $[c, d]$

revolve around y-axis

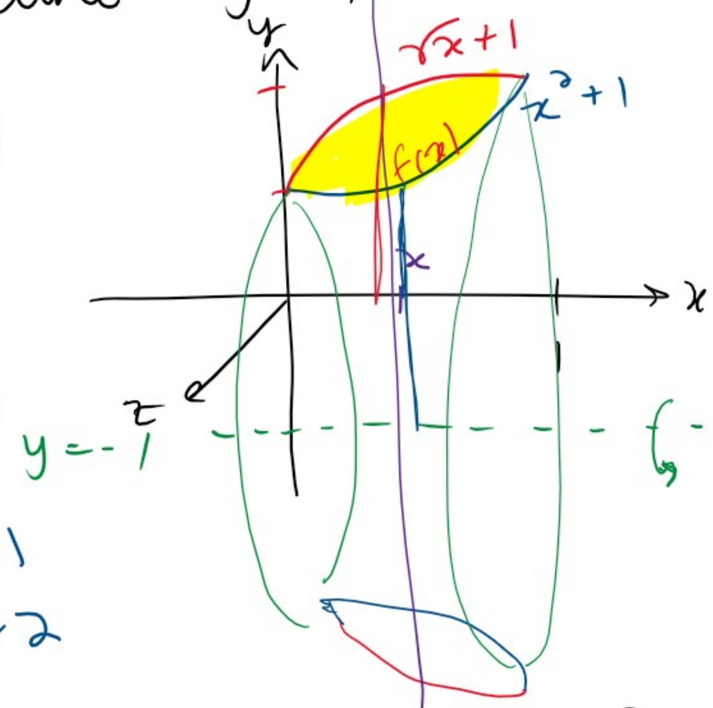


$$V = \int_c^d \pi \left([p(y)]^2 - [q(y)]^2 \right) dy$$

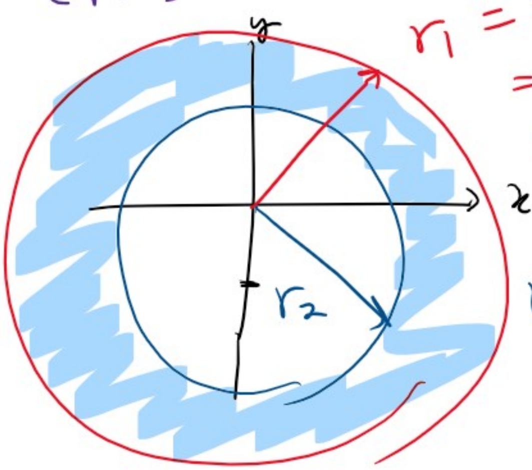
NOTE: We can revolve around lines <u>other</u> than the axes

$f(x) = \sqrt{x} + 1$ $\quad g(x) = x^2 + 1$ $\quad [0, 1]$

revolve around the line $y = -1$



cross-section



$r_1 = f(x) - (-1)$
$= \sqrt{x} + 1 + 1$
$= \sqrt{x} + 2$

$r_2 = g(x) + 1$
$= x^2 + 1 + 1$
$r_2 = x^2 + 2$

$$A(x) = \pi r_1^2 - \pi r_2^2 = \pi (\sqrt{x} + 2)^2 - \pi (x^2 + 2)^2$$

$$V = \int_0^1 \left[\pi (\sqrt{x} + 2)^2 - \pi (x^2 + 2)^2 \right] dx$$

$$\int_0$$

$$= \pi \int_0^1 \left[x + 4\sqrt{x} \cancel{+4} + \left(-x^4 \mp 4x^2 \mp \cancel{4}\right)\right] dx$$

$$= \pi \int_0^1 \left[x + 4\sqrt{x} - x^4 - 4x^2 \right] dx$$

$$= \pi \left[\frac{x^2}{2} + \frac{4x^{3/2}}{3/2} - \frac{x^5}{5} - \frac{4x^3}{3} \right]_0^1$$

$$= \pi \left[\frac{1}{2} + \frac{8}{3} - \frac{1}{5} - \frac{4}{3} - 0 \right] = \boxed{\frac{49\pi}{30}}$$