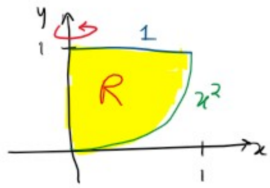
6.4: Volumes by Shells

Announcements:
- REC 229, 236, 243 – new TA Nikhil Mehra – mehran@purdue.edu
- if you are in PHYS 172 – 008 – 095 email Dr. Hood about exam conflicts

GOALS: – use Shell Method to find V

WARM UP: Let R be the region bounded by $y = x^2$ and $y = 1$. Revolve R around the y-axis. Use the Washer Method to set up the integral for the volume V

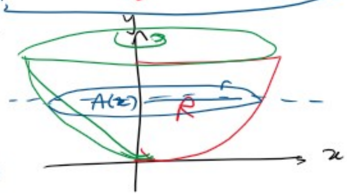


(a) $V = \int_0^1 \pi\left[1 - x^4\right] dx$ $\qquad$ $\boxed{\text{(b) } V = \int_0^1 \pi y \, dy}$

$A(y) = \pi\left(f(y)\right)^2 = \pi\left[y^{1/2}\right]^2$

revolve around y-axis

Washer $\Rightarrow$ integrate wrt y

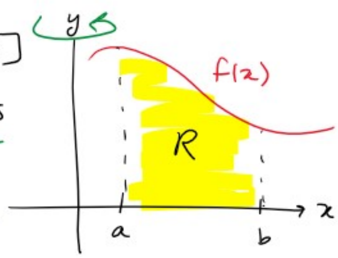


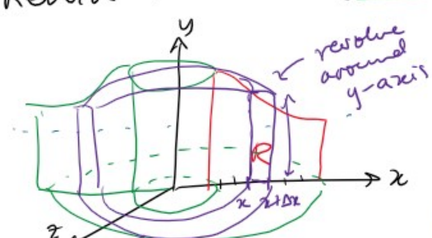
$$\boxed{V = \int_0^1 \pi y \, dy}$$

I. Volume by Shells:

function $f(x)$ on $[a, b]$

Revolve R around y-axis





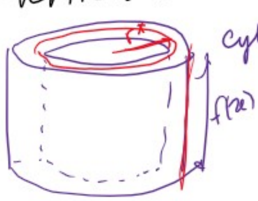
solid of revolution

(Washer – horizontal slice

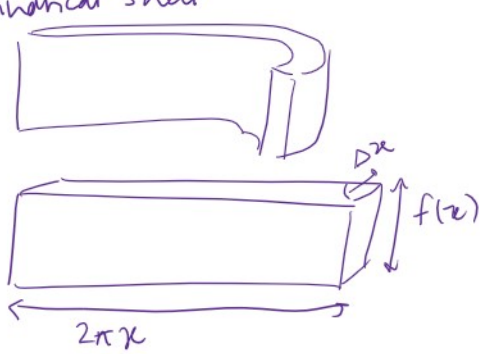
)

Today – vertical slice

Shell Method



cylindrical shell



volume of cylindrical shell

$= (2\pi x) f(x) \Delta x$

Sum up over all x_k

$$V \approx \sum_{k=1}^{n} 2\pi x_k f(x_k) \Delta x$$

$\longrightarrow$ limit as $n \to \infty$

Shell

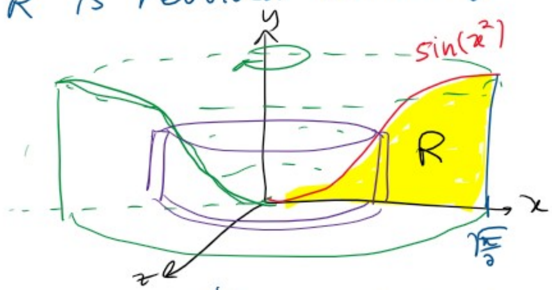
$k=1$

$\longrightarrow$ limit as $n \to \infty$

$$\boxed{V = \int_a^b 2\pi x\, f(x)\, dx}$$ Shell Method

Ex: A sine bowl

Let R be the region bounded by
$f(x) = \sin(x^2)$
the x-axis, and line $x = \sqrt{\frac{\pi}{2}}$
Find the volume of the solid of revolution when R is revolved around y-axis



Shell Method: $V = \int_a^b \underbrace{2\pi x}_{\text{shell circumference}} \underbrace{f(x)}_{\text{shell hight}} dx$

$$= \int_0^{\sqrt{\frac{\pi}{2}}} 2\pi x \sin(x^2)\, dx$$

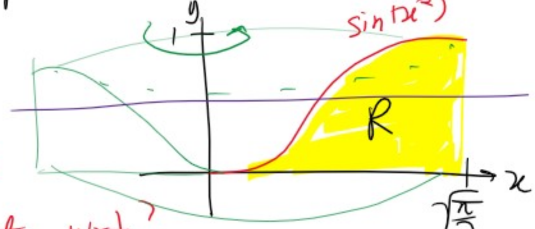
u-substitution
let $u = x^2$ $\qquad \sin(x^2) \longrightarrow \sin(u)$
$du = 2x\,dx$ $\qquad 2\pi x\,dx \to \pi\,du$

$x = 0 \longrightarrow u = 0$
$x = \sqrt{\frac{\pi}{2}} \longrightarrow u = \frac{\pi}{2}$

$$V = \pi \int_0^{\frac{\pi}{2}} \sin(u)\,du = \pi \Big[-\cos(u)\Big]_0^{\frac{\pi}{2}}$$

$$= \pi\left[-\cancelto{0}{\cos\left(\tfrac{\pi}{2}\right)} + \cancelto{1}{\cos(0)}\right] = \boxed{\pi = V}$$

Q: How does compare to the washer method?



POLL: For the washer method, which variable should we integrate wrt?

$\boxed{\text{(b) wrt } y}$

Which ... should we integrate wrt?

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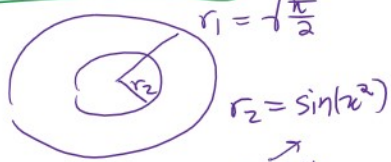
(a) wrt x

$\boxed{\text{(b) wrt } y}$



horizontal cross-section
washer — $\perp$ to y-axis
$\Rightarrow$ integrate along y.

inner radius $r_2 = \left[\sin^{-1}(y)\right]^{1/2}$

$y = \sin(x^2)$

$\sin^{-1}(y) = x^2 \longrightarrow x = \left[\sin^{-1}(y)\right]^{1/2}$

$r_1 = \sqrt{\frac{\pi}{2}}$

Washer: $V = \int_c^d \pi\left[r_1^2 - r_2^2\right] dy$

$$\boxed{V = \int_0^1 \pi\left[\frac{\pi}{2} - \sin^{-1}(y)\right] dy}$$

What is $\int_0^1 \sin^{-1}(y)\,dy$?

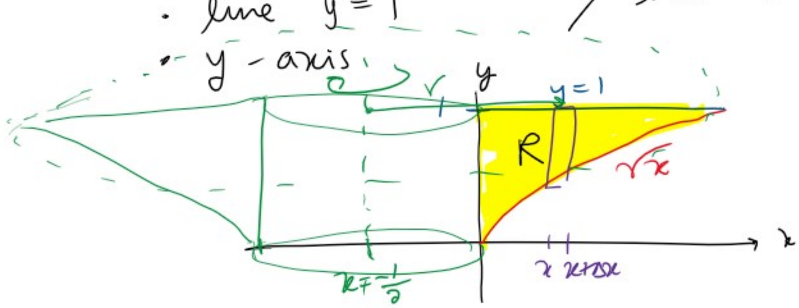
We will cover in Lesson 10 — Integration by Parts

II. Revolving about other lines:

Let R be the region bounded by

- curve $y = \sqrt{x}$
- line $y = 1$
- y-axis

revolve around line $x = -\frac{1}{2}$

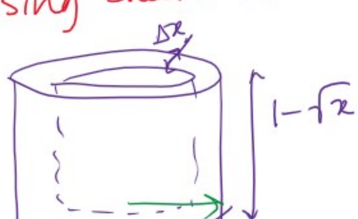


Shell Method — take a slice in x

POLL: Set up the integral for V using Shell Method

(a) $V = \int_0^1 2\pi\left(x+\frac{1}{2}\right)(1-\sqrt{x})\,dx$

(b) $V = \int_0^1 2\pi x\,(1-\sqrt{x})\,dx$



(b) $V = \int_0^1 2\pi x\,(1-\sqrt{x})\,dx$

(c) $V = \int_0^1 2\pi x \sqrt{x}\,dx$

$r = x + \frac{1}{2}$

$$V = \int_0^1 \underbrace{2\pi\left(x+\tfrac{1}{2}\right)}_{\text{circumf.}} \underbrace{\left(1-\sqrt{x}\right)}_{\text{height}} dx$$

Q: When do you use Shell Method vs. Washer Method?

A: In principle: either method works

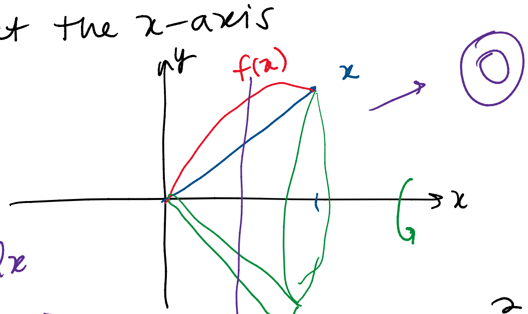
In practice: one method produces an integral that's easier to evaluate

Ex: Set up the integrals for both

- Washer Method
- Shell Method

Region R bounded by $f(x) = 2x - x^2$, $g(x) = x$ on $[0,1]$

Revolve R about the x-axis



Washer Method:

Vertical slice

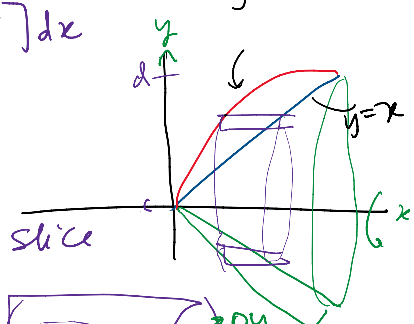
$$V = \int_a^b \pi\left[\underbrace{f(x)}_{\text{outer } r}{}^2 - \underbrace{g(x)}_{\text{inner}}{}^2\right]dx$$

$$= \int_0^1 \pi\left[(2x-x^2)^2 - [x]^2\right] dx$$

$$= \int_0^1 \pi\left[4x^2 - 4x^3 + x^4 - x^2\right] dx$$

$$V = \pi\int_0^1 (3x^2 - 4x^3 + x^4)\,dx$$

$y = 2x - x^2$

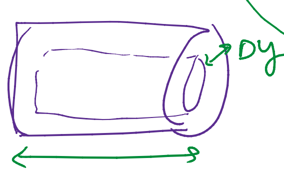


Washer	Shell
revolve around x	revolve around x
integrate wrt x	integrate wrt y

Shell Method:

integrate wrt y

horizontal slice



height

$$\left[y - \left(1 - \sqrt{1-y}\right)\right]$$

$y = 2x - x^2$

$x^2 - 2x + y = 0$

~~quadratic formula~~

$[y-(1 \pm \sqrt{1-y}))]$

radius $r = y$

$$\boxed{V = \int_0^1 2\pi y\left[y-(1-\sqrt{1-y})\right]dy}$$

$\cancel{x^2 - 2x + y = 0}$

~~quadratic formula~~

$\cancel{x = \frac{+2}{2} \pm \frac{\sqrt{(2)^2 - 4\cdot 1\cdot (y)}}{2}}$

$\cancel{x = 1 \pm \sqrt{1-y}}$

corrections after class

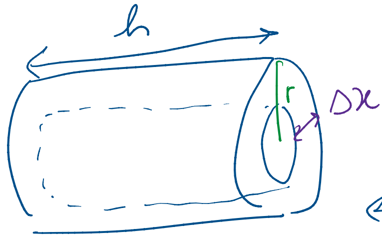
$y = 2x - x^2$

$x^2 - 2x + y = 0$

quadratic formula

$x = \frac{+2}{2} \pm \frac{\sqrt{(-2)^2 - 4\cdot 1\cdot y}}{2}$

$x = 1 \pm \sqrt{1-y}$



Should we use + or − root?

@ $y=0$, $x=0$

so use − root

$x = 1-\sqrt{1-y}$

$$h = f(y) - g(y) = y - (1-\sqrt{1-y})$$

$$h = y - 1 + \sqrt{1-y}$$

$$r = y$$

So the shell method gives

$$V = \int_0^1 \underbrace{(2\pi y)}_{\text{shell circumf.}}\underbrace{\left[y-1+\sqrt{1-y}\right]}_{\text{shell height}}dy$$