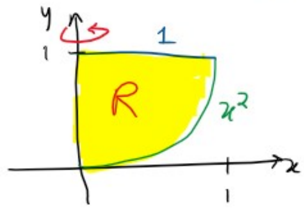
6.4: Volumes by Shells

Announcements:

- if you are in PHYS 172 − 008 / − 095 email Dr. Hood about exam conflicts

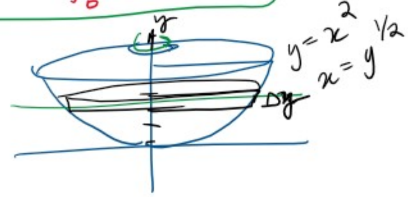
GOALS: - use Shell Method to find V

WARM UP: Let R be the region bounded by $y=x^2$ and $y=1$. Revolve R around the y-axis. Use the Washer Method to set up the integral for the volume V



(a) $V=\int_0^1 \pi\left[1-x^4\right]dx$ $\qquad$ $\boxed{\text{(b) } V=\int_0^1 \pi y\, dy}$

horizontal slice ($\perp$ to axis of rotation)



A —  $r=y^{1/2}$

$A(y)=\pi r^2=\pi\left(y^{1/2}\right)^2=\pi y$

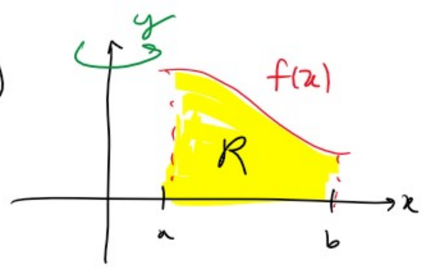
$$\boxed{V=\int_0^1 \pi y\, dy}$$

Washer Method if revolve around the y-axis integrate wrt y-axis

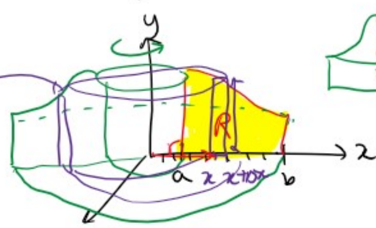
I. Volumes by Shells:

function $f(x)$ on $[a,b]$

Revolve region R around the y-axis



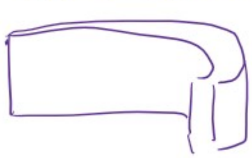
solid of revolution



vertical slice

cylindrical shell

cut + flatten out



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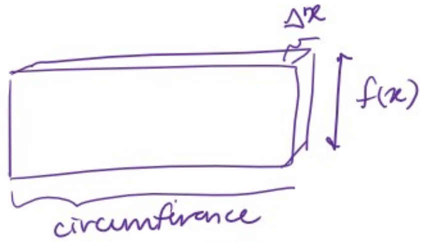


circumference

$2\pi r = 2\pi x$

Volume of shell

$V \sim (2\pi x)f(x)\Delta x$

Sum up over all x_k

$V \approx \sum^{n} 2\pi x_k\, f(x_k)\Delta x$

Sum up over all x_k

$$V \approx \sum_{k=1}^{n} 2\pi x_k f(x_k) \Delta x$$

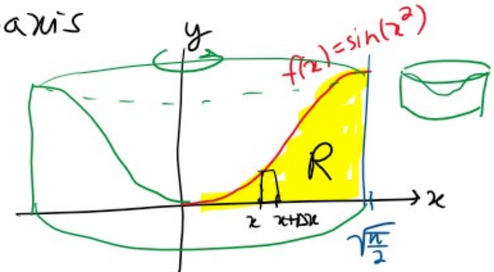
$\longrightarrow$ limit as $n \to \infty$

$$\boxed{V = \int_a^b 2\pi x f(x)\, dx}$$ The shell method

$f(x)$ is continuous $[a,b]$
revolve region R around the y-axis

<u>NOTE</u>: Shell Method
revolve around the y-axis
integrate wrt x

<u>Ex</u>: A sine bowl
Let R be the region bounded by
$f(x) = \sin(x^2)$, x-axis, $x = \sqrt{\frac{\pi}{2}}$
Find the volume V when R is revolved around the y-axis



<u>Shell Method</u>

$$V = \int_a^b \underbrace{2\pi x}_{\text{circumf.}} \underbrace{f(x)}_{\text{height}}\, dx$$

$$= \int_0^{\sqrt{\frac{\pi}{2}}} 2\pi x \sin(x^2)\, dx$$

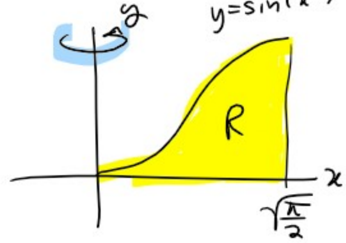
u-substitution
$u = x^2$
$du = 2x\,dx$
when $x=0$, $u=0$
$x = \sqrt{\frac{\pi}{2}}$, $u = \frac{\pi}{2}$

$$= \int_0^{\frac{\pi}{2}} \pi \sin(u)\, du$$

$$= \pi \Big[-\cos(u)\Big]_0^{\frac{\pi}{2}} = \pi\Big[-\cancel{\cos\left(\tfrac{\pi}{2}\right)}^{0} + \cancel{\cos(0)}^{1}\Big]$$

$$\boxed{V = \pi}$$

Q: How does this compare to the Washer Method?

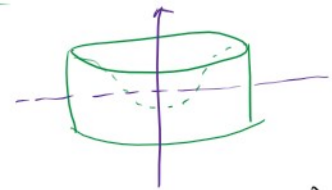


<u>POLL</u>: Which variable should we integrate with respect to? (Washer)

(a) wrt x

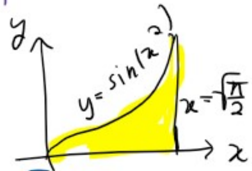
(b) wrt y

Washer cross-sections



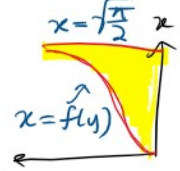
horizontal

integrate wrt y



$$V=\int_c^d \pi\left[f(y)^2-g(y)^2\right]dy$$

$f(y)=\sqrt{\frac{\pi}{2}}$



$y=\sin(x^2)$

$\sin^{-1}(y)=x^2$

$x=\left[\sin^{-1}(y)\right]^{1/2}=g(y)$

$$V=\int_0^1 \pi\left[\frac{\pi}{2}-\underbrace{\sin^{-1}(y)}\right]dy$$

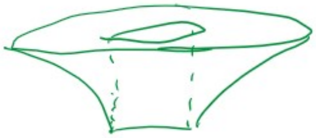
What is $\int_0^1 \sin^{-1}(y)\,dy$?

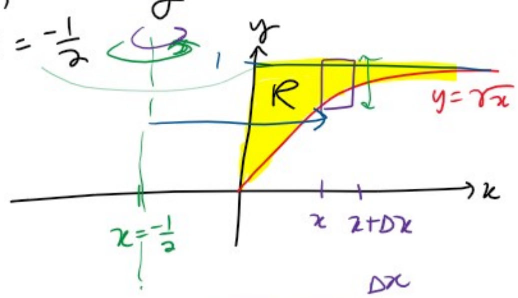
We will cover this in Lesson 10

II. Revolving about other lines:

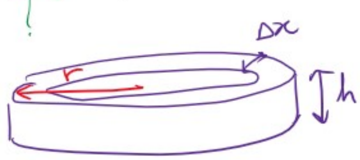
Let R be the region bounded by $y=\sqrt{x}$, $y=1$, and y-axis

revolve around $x=-\frac{1}{2}$





Shell Method



$h=1-\sqrt{x}$

$r=x+\frac{1}{2}$

$2\pi r h\,\Delta x$

$$\int_0^1 2\pi\left(x+\tfrac{1}{2}\right)\left(1-\sqrt{x}\right)dx$$

POLL:

Set up the integral using the shell method

(a) $V=\int_0^1 2\pi\left(x+\frac{1}{2}\right)\left(1-\sqrt{x}\right)dx$

(b) $V=\int_0^1 2\pi x\left(1-\sqrt{x}\right)dx$

(c) $V=\int_0^1 2\pi x\sqrt{x}\,dx$

Q: When do you use Washer vs. Shell method?

Q: When do you use Washer vs. Shell method?

A: In principle: either works

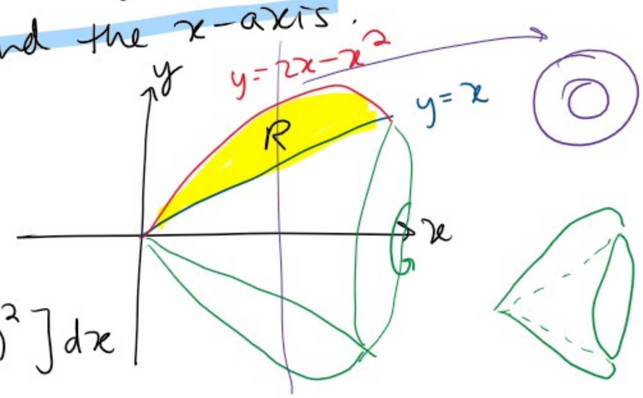
In practice: one has an easier integral

Ex: Set up the integrals for Washer + Shell

Region R bounded by

$f(x) = 2x - x^2$, $g(x) = x$ on $[0,1]$

Revolve R around the x-axis.



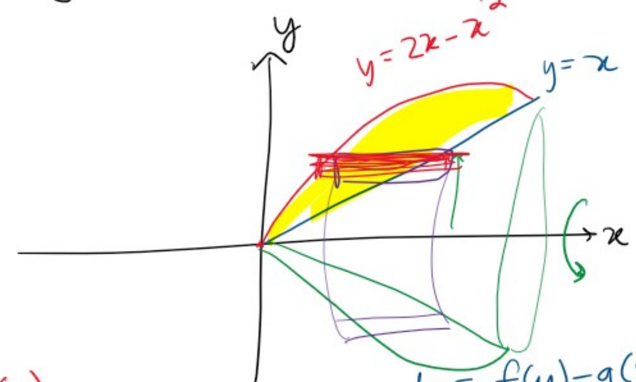
Washer Method

revolve x-axis

integrate wrt x

$$V = \int_0^1 \pi\left[(2x-x^2)^2 - (x)^2\right]dx$$

$$= \int_0^1 \pi\left[3x^2 - 4x^3 + x^4\right]dx$$



Shell Method:

revolve around x

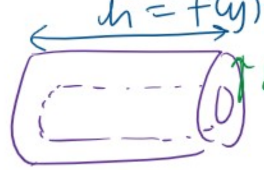
→ integrate wrt y

$y = x \rightarrow x = y$

$y = 2x - x^2 \rightarrow x = g(y)$

$h = f(y) - g(y) = y - (1 - \sqrt{1-y})$

cylindrical shell



$r = y$

$x^2 - 2x + y = 0$

quadratic formula

$$x = \frac{+2}{2} \pm \frac{\sqrt{(-2)^2 - 4 \cdot 1 \cdot y}}{2}$$

$$= 1 \pm \sqrt{1-y}$$

$\xrightarrow{\text{neg root}}$

$x = 1 - \sqrt{1-y} = g(y)$

(bc @ $y=0 \rightarrow x=0$)

$$V = \int_0^1 (2\pi y)\left[y - 1 + \sqrt{1-y}\right)dy$$

$$V = \int_0^1 \underbrace{(2\pi y)}_{\text{circumf.}} \underbrace{\left[y - 1 + \sqrt{1-y} \right)}_{\text{height}} \, dy$$