Lesson 8 - MA 166 LEC 200

6.5 - Length of Curves

6.6 - Surface Area

Announcements:

- Exam 1 on Wed Feb 9 @ 6:30pm
- study guide + FAQs

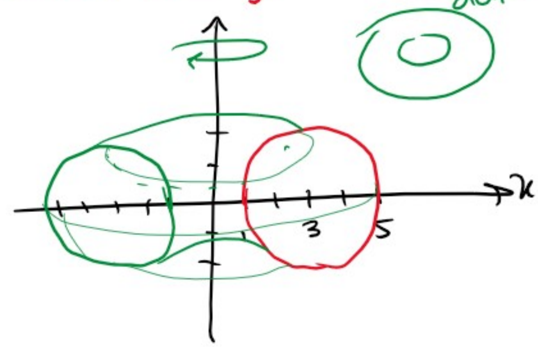
Shell Method

GOALS:

- Find arc lengths
- Find surface area of curves revolved around a given axis

WARM UP: A torus is formed when a circle of $r=2$, centered at $(3,0)$ is revolved around the y-axis.

3D donut



The Shell method says:

$$V = \int_1^5 2\pi x\, f(x)\,dx \text{, where}$$

$$\boxed{(a)\quad f(x) = 2\sqrt{4-(x-3)^2}}$$

$$(b)\quad f(x) = 2\sin\left(\frac{\pi(x-1)}{4}\right)$$

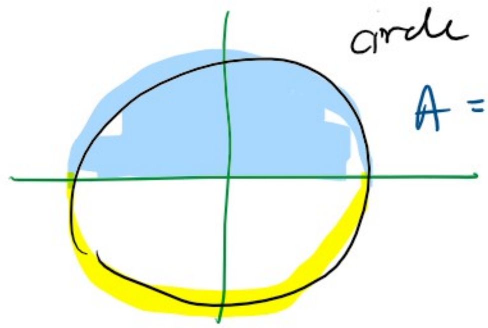
$r = 2$

center $(3,0)$

$$(x-3)^2 + y^2 = 2^2$$

$$y = \pm\sqrt{4-(x-3)^2}$$

circle



$$A = 2\times\left(\begin{matrix}\text{blue}\\\text{region}\end{matrix}\right)$$

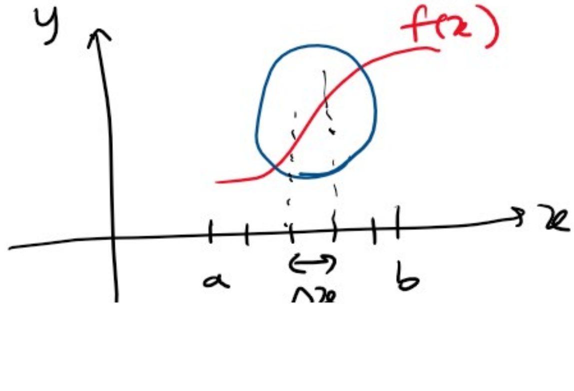
Symmetry $f(x) = 2\sqrt{4-(x-3)^2}$

I. Length of Curves:

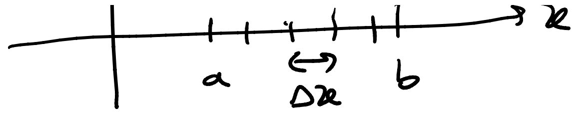
Given a function $y = f(x)$ on $[a,b]$

Find the length L of the curve $f(x)$

arc length



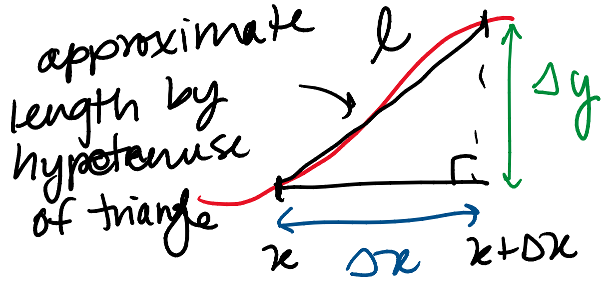
of the curve.



arc length

Slice up the x-axis

$$l = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$



$$L \approx \sum_{k=1}^{n} \sqrt{\frac{(\Delta x)^2 + (\Delta y_k)^2}{(\Delta x)^2}}\ \Delta x \qquad \text{pull out a } \Delta x$$

$$= \sum_{k=1}^{n} \sqrt{1 + \frac{(\Delta y_k)^2}{(\Delta x)^2}}\ \Delta x$$

$$= \sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x}\right)^2}\ \Delta x$$

limit as $n \to \infty$

$y = f(x)$

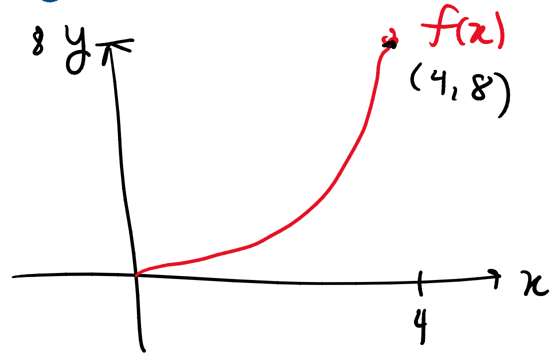
$$\boxed{\begin{aligned} L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx \\ &= \int_a^b \sqrt{1 + \{f'(x)\}^2}\ dx \end{aligned}}$$

Arc length $y = f(x)$

Need $f(x)$ to have a continuous derivative on $[a,b]$

Ex: Find the length of the curve $f(x) = x^{3/2}$

Ex: Find the length of the curve $f(x) = x^{\dots}$ between $x=0$ and $x=4$



$$y = f(x) = x^{3/2}$$

$$f'(x) = \frac{3}{2}x^{1/2}$$

$$L = \int_a^b \sqrt{1 + \{f'(x)\}^2}\, dx$$

$$= \int_0^4 \sqrt{1 + \left\{\frac{3}{2}x^{1/2}\right\}^2}\, dx = \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

u-substitution $\quad u = 1 + \frac{9x}{4} \quad$ @ $x=0 \quad u = 1$

$du = \frac{9}{4}dx \quad$ @ $x=4 \quad u = 1 + \frac{9 \cdot \cancel{4}}{\cancel{4}}$

$u = 10$

$$= \int_1^{10} u^{1/2}\left(\frac{4}{9}du\right) = \frac{4}{9}\int_1^{10} u^{1/2}du$$

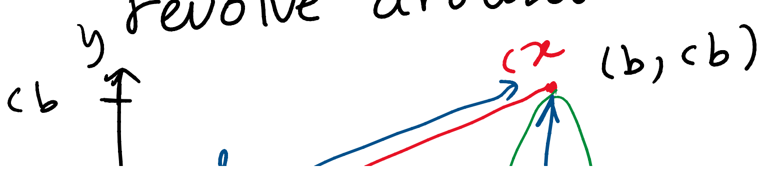
$$= \frac{4}{9}\left[\frac{u^{3/2}}{3/2}\right]_1^{10} = \frac{4}{9}\left(\frac{2}{3}\right)\left[10^{3/2} - 1^{3/2}\right]$$

$$= \boxed{\frac{8}{27}\left[10\sqrt{10} - 1\right] = L}$$

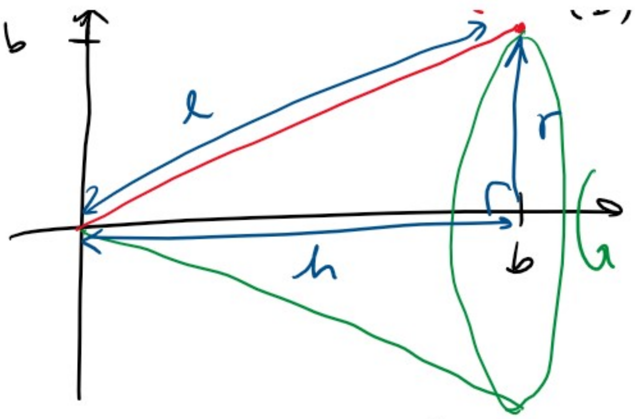
II. Surface Area:

$f(x) = cx$ with $c > 0$ on $[0, b]$ $\quad b > 0$

revolve around x-axis



Cone



cone

height $h = b$

baseradius $r = cb$

slant height l

$$l = \sqrt{h^2 + r^2}$$

$$= \sqrt{b^2 + (cb)^2}$$

$$= b\sqrt{1 + c^2}$$

surface area of cone

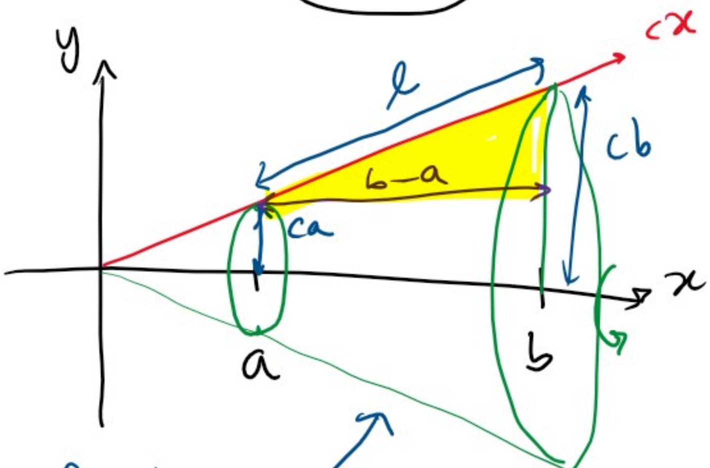
$$S = \pi r l$$

$$= \pi (cb)\left[b\sqrt{1+c^2}\right]$$

$$= \pi c b^2 \sqrt{1+c^2}$$

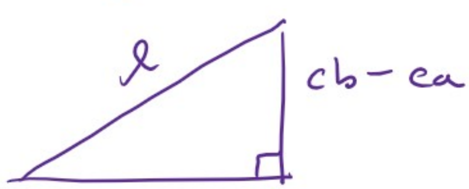
<u>Def</u>: The <u>frustrum</u> is a cone with the top chopped off

frustrum lamp shade



frustrum

Q: What is the slant hight l?



$$l = \sqrt{(b-a)^2 + (cb-ca)^2}$$

$$= (b-a)\sqrt{1+c^2}$$

Surface area of the frustrum

$$\ldots b^2\sqrt{1+c^2} \quad \pi c a^2\sqrt{1+c^2}$$

the frustrum

$$S_f = \underbrace{S_b}_{\substack{\text{cone}\\ h=b}} - \underbrace{S_a}_{\substack{\text{cone}\\ h=a}} = \pi c b^2\sqrt{1+c^2} - \pi c a^2\sqrt{1+c^2}$$

$$= \pi c\,(b^2-a^2)\sqrt{1+c^2}$$

$$= \pi c\,(b+a)\underbrace{(b-a)\sqrt{1+c^2}}_{\ell}$$

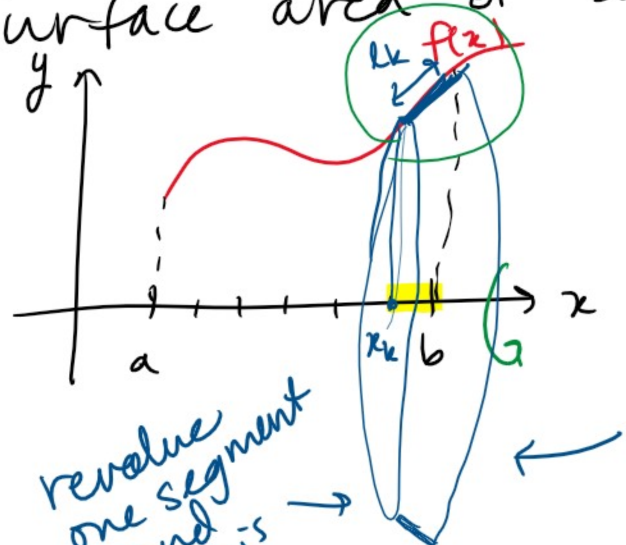
$$= \pi c\,(b+a)\,\ell = \pi(\underbrace{cb}_{f(b)} + \underbrace{ca}_{f(a)})\,\ell$$

$$\boxed{S_f = \pi\left(f(b)+f(a)\right)\ell}$$

Surface area of solids of revolution

$f(x)$ on $[a,b]$

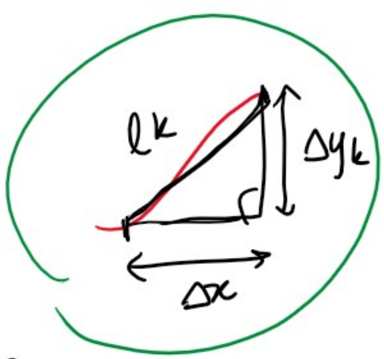
revolve around x-axis



revolve one segment around x-axis

approximate by a frustrum

$$S_k \approx \pi\left[f(x_k) + f(x_k+\Delta x)\right] l_k$$



$$l_k \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$S \approx \sum^{n} \pi\left[f(x_k) + f(x_k+\Delta x)\right]\sqrt{(\Delta x)^2 + (\Delta y_k)^2}$$

$$S \approx \sum_{k=1}^{n} \pi \left[f(x_k) + f(x_k + \Delta x) \right] \sqrt{(\Delta x)^2 + (\Delta y_k)^2}$$

$$= \sum_{k=1}^{n} \pi \left[f(x_k) + f(x_k + \Delta x) \right] \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$$

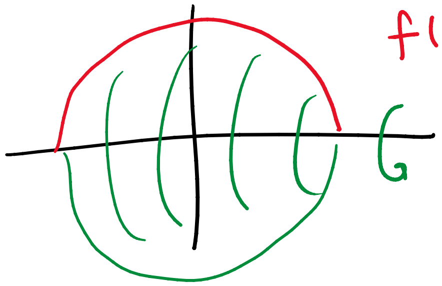
limit $n \longrightarrow \infty$

$$S = \int_a^b \pi \left(2 f(x)\right) \sqrt{1 + \{f'(x)\}^2}\, dx$$

$$\boxed{S = \int_a^b 2\pi f(x) \sqrt{1 + \{f'(x)\}^2}\, dx}$$

Surface Area

Ex: Surface area of unit sphere



$f(x) = +\sqrt{1 - x^2}$

revolve around x-axis

$$f'(x) = \frac{1}{2}(1 - x^2)^{-1/2}(-2x)$$

$$= \frac{-x}{\sqrt{1 - x^2}}$$

$$S = \int_{-1}^{1} 2\pi \sqrt{1 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{1 - x^2}}\right)^2}\, dx$$

$$= \int_{-1}^{1} 2\pi\sqrt{1-x^2}\,\sqrt{\frac{1-\cancel{x^2}+\cancel{x^2}}{1-x^2}}\,dx$$

$$= \int_{-1}^{1} 2\pi\,\frac{\cancel{\sqrt{1-x^2}}}{\cancel{\sqrt{1-x^2}}}\,dx = \left[2\pi x\right]_{-1}^{1} = \boxed{4\pi = S}$$

$$\boxed{S = 4\pi r^2}$$