Lesson 8 - MA 166 LEC 286

6.5 - Length of Curves

6.6 - Surface Area

Announcements:

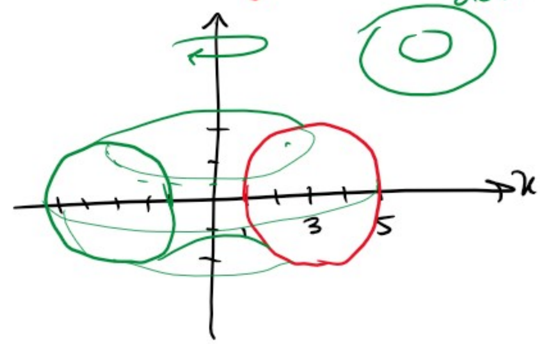
- Exam 1 on Wed Feb 9 @ 6:30pm
- study guide + FAQs

Shell Method

GOALS:

- Find arc lengths
- Find surface area of curves revolved around a given axis

WARM UP: A torus is formed when a circle of $r=2$, centered at $(3,0)$ is revolved around the y-axis. 3D donut



The Shell method says:

$$V = \int_1^5 2\pi x\, f(x)\,dx, \text{ where}$$

(a) $f(x) = 2\sqrt{4-(x-3)^2}$

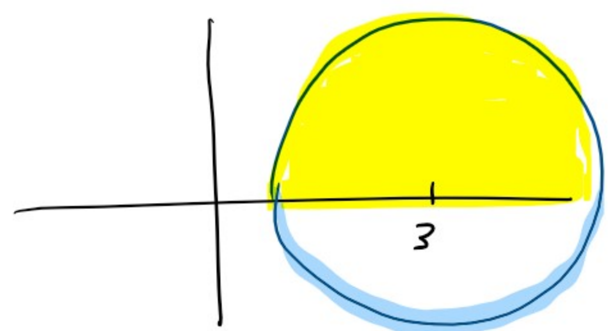
(b) $f(x) = 2\sin\left(\frac{\pi(x-1)}{4}\right)$

$r = 2$

center $(3,0)$

$(x-3)^2 + y^2 = 2^2$

$y = \pm\sqrt{4-(x-3)^2}$

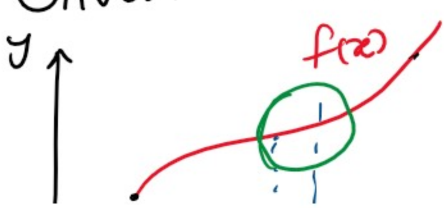


Area of circle $= 2\times$

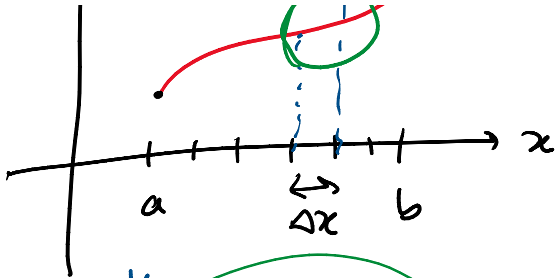
$f(x) = 2\sqrt{4-(x-3)^2}$

I. Length of Curves:

Given a function $y = f(x)$ on $[a,b]$



Find the arc length L of the curve $f(x)$

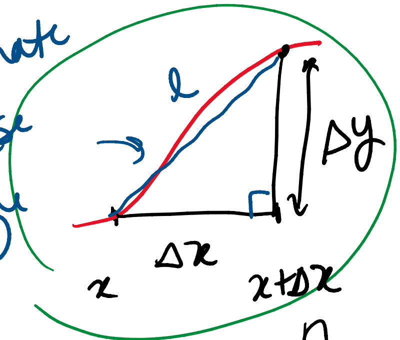


of the curve $f(x)$

slice up the x-axis

$$\ell = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

approximate by hypotenuse of triangle



pull out a Δx

$$L \approx \sum_{k=1}^{n} \sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta x)^2}}\ \Delta x$$

$$= \sum_{k=1}^{n} \sqrt{1 + \frac{(\Delta y)^2}{(\Delta x)^2}}\ \Delta x$$

$$= \sum_{k=1}^{n} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

limit as $n \to \infty$ $(\Delta x \to 0)$

$$\boxed{\begin{aligned} L &= \int_a^b \sqrt{1 + \left\{\frac{dy}{dx}\right\}^2}\ dx \\ &= \int_a^b \sqrt{1 + \{f'(x)\}^2}\ dx \end{aligned}}$$

$y = f(x)$

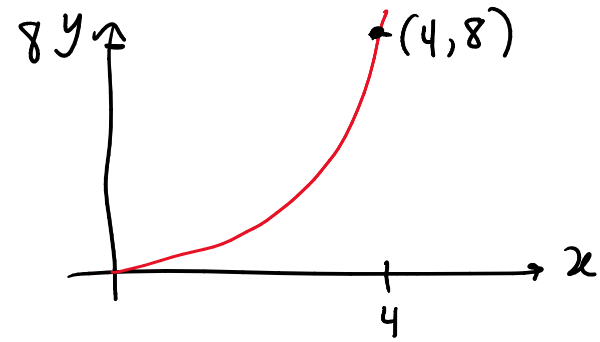
Arc Length for $y = f(x)$

Need $f(x)$ to have a continuous first derivative on $[a,b]$

$f(x) = x^{3/2}$

first derivative

Ex: Find the length of the curve $f(x) = x^{3/2}$ between $x = 0$ and $x = 4$



$$f(x) = x^{3/2}$$

$$f'(x) = \frac{3}{2} x^{1/2}$$

$$L = \int_a^b \sqrt{1 + \{f'(x)\}^2}\, dx$$

$$= \int_0^4 \sqrt{1 + \left(\frac{3}{2} x^{1/2}\right)^2}\, dx = \int_0^4 \sqrt{1 + \frac{9x}{4}}\, dx$$

u-substitution

$$u = 1 + \frac{9x}{4} \qquad @\, x = 0 \quad u = 1$$

$$du = \frac{9\,dx}{4} \qquad @\, x = 4 \quad u = 1 + \frac{9 \cdot 4}{4} = 10$$

$$= \int_1^{10} u^{1/2} \left(\frac{4\,du}{9}\right) = \frac{4}{9} \int_1^{10} u^{1/2}\, du$$

$$= \frac{4}{9} \left[\frac{u^{3/2}}{3/2}\right]_1^{10} = \frac{4}{9} \cdot \frac{2}{3} \left[10^{3/2} - 1^{3/2}\right]$$

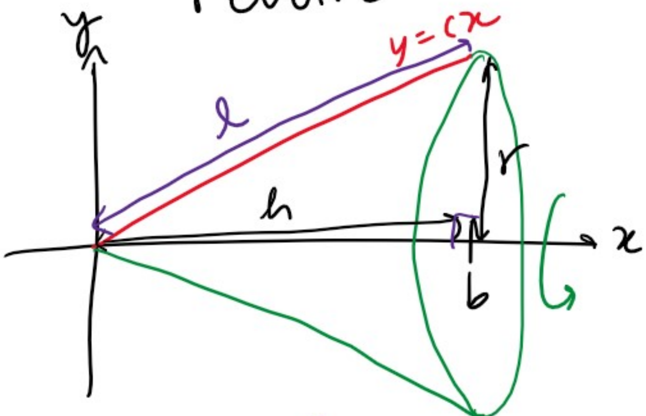
$$= \boxed{\frac{8}{27}\left[10\sqrt{10} - 1\right] = L}$$

II. Surface Area:

$f(x) = cx$ with $c > 0$ on $[0, b]$

revolve around x-axis

$f(x) = cx$ with c ...
revolve around x-axis



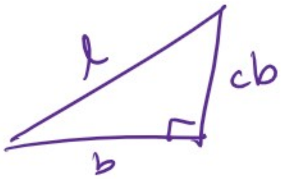
cone

height $h = b$

base radius $r = cb$

slant length $l =$

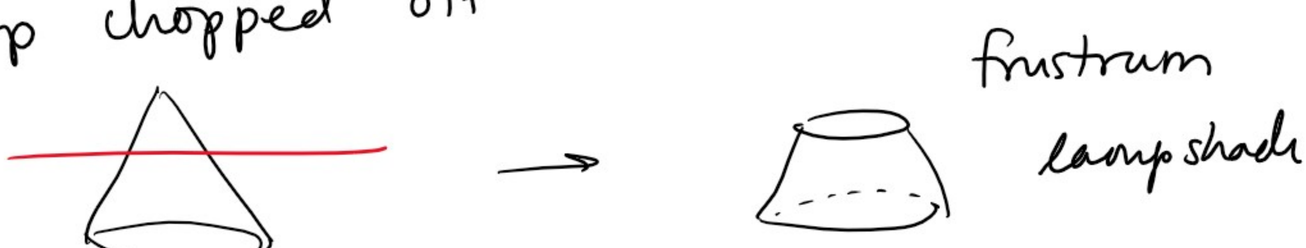
$$l = \sqrt{b^2 + (cb)^2}$$
$$= b\sqrt{1 + c^2}$$

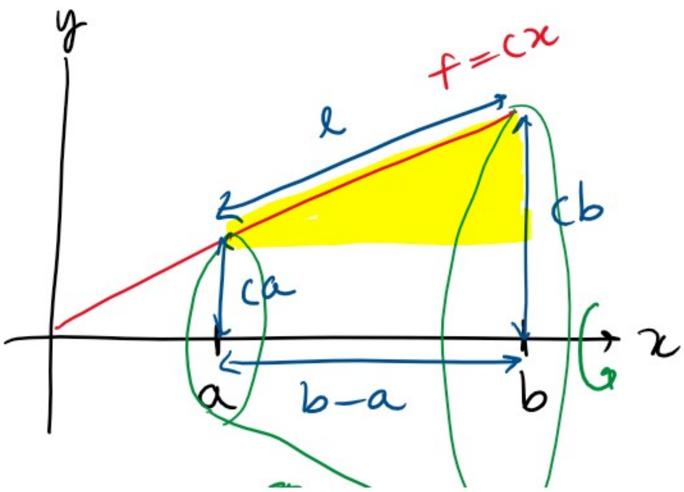


Surface area of the cone

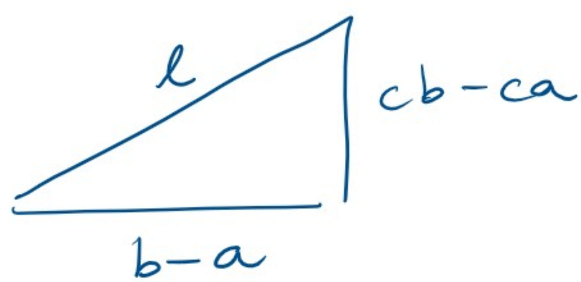
$$S = \pi r l = \pi (cb)\left(b\sqrt{1+c^2}\right)$$
$$S = \pi c b^2 \sqrt{1+c^2}$$

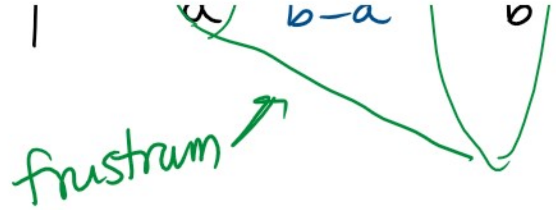
__Def__: The __frustrum__ is a cone with the top chopped off





$$l = \sqrt{(b-a)^2 + (cb-ca)^2}$$





$$\frac{\quad}{b-a}$$

$$\ell = (b-a)\sqrt{1+c^2}$$

Surface area of frustrum

$$S_f = \underbrace{S_b}_{\substack{\text{cone} \\ h=b}} - \underbrace{S_a}_{\substack{\text{cone} \\ h=a}} = \pi c b^2\sqrt{1+c^2} - \pi c a^2\sqrt{1+c^2}$$

$$= \pi c\,(b^2-a^2)\sqrt{1+c^2}$$

$$= \pi c\,(b+a)(b-a)\sqrt{1+c^2}$$

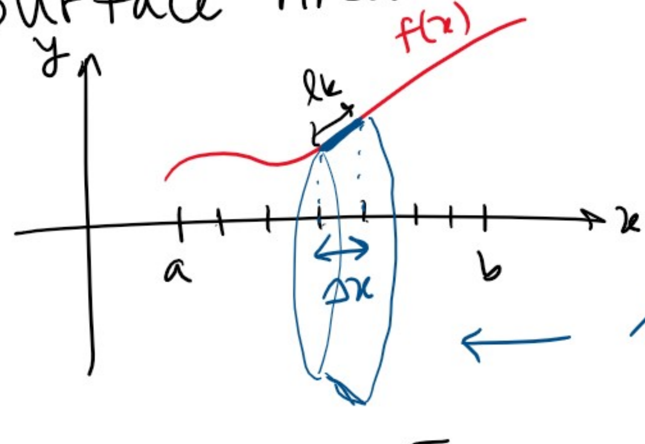
$$= \pi c\,(b+a)\,\ell = \pi\,(\underbrace{cb}_{f(b)} + \underbrace{ca}_{f(a)})\,\ell$$

$$\boxed{S_f = \pi\,(f(b)+f(a))\,\ell}$$

Surface Area of Solid of revolution

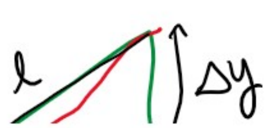
$f(x)$ on $[a,b]$

revolve around the x-axis



looks like a frustrum

$$S_k \approx \pi\left[f(x_k) + f(x_k+\Delta x)\right]\ell_k$$



$$\ell_k \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

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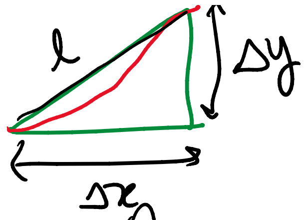


$$S \approx \sum_{k=1}^{n} \pi \left[f(x_k) + f(x_k + \Delta x) \right] \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

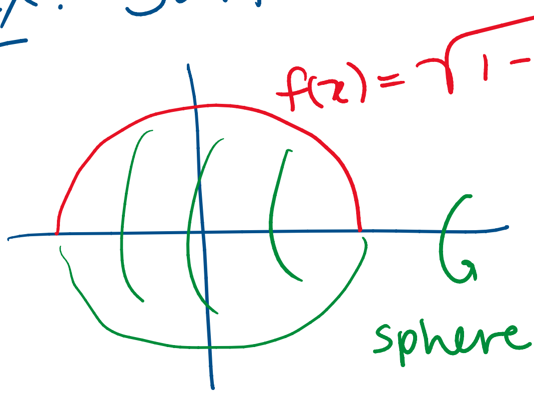
$$= \sum_{k=1}^{n} \pi \left[f(x_k) + f(x_k + \Delta x) \right] \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$$

$\longrightarrow$ limit $n \to \infty$, $\Delta x \to 0$

$$S = \int_a^b \pi \left(2 f(x) \right) \sqrt{1 + \{f'(x)\}^2}\, dx$$

$$\boxed{S = \int_a^b 2\pi f(x) \sqrt{1 + \{f'(x)\}^2}\, dx}$$

Ex: Surface area of the unit sphere



$$f'(x) = \frac{1}{2} (1 - x^2)^{-1/2} (-2x)$$

$$= \frac{-x}{\sqrt{1 - x^2}}$$

$$S = \int_a^b 2\pi f(x) \sqrt{1 + \{f'(x)\}^2}\, dx$$

$$= \int_{-1}^{1} 2\pi \sqrt{1 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{1 - x^2}} \right)^2}\, dx$$

$$= \int_{-1}^{1} 2\pi \cancel{\sqrt{1-x^2}} \sqrt{\frac{1-\cancel{x^2}+\cancel{x^2}}{\cancel{1-x^2}}}\, dx$$

$$= \int_{-1}^{1} 2\pi \, dx = \left[2\pi x\right]_{-1}^{1} = \boxed{4\pi = S}$$

$$S = 4\pi r^2$$