Lesson 9 - MA 166 LEC 200

6.7 Physical Applications

Announcements:

- exam 1 on Wed Feb 9 @ 6:30pm
- any one can attend in-person lectures now

GOALS:

- Find mass of thin bars
- Solve applications involving work
- Solve applications involving pressure + hydrostatic force

WARM UP: Express the function $y = f(x) = \ln(x + \sqrt{x^2-1})$ as a function in y, i.e. $x = g(y)$

$\boxed{(a)\ x = \frac{e^y + e^{-y}}{2}}$ (b) $x = \frac{e^{2y}+1}{2}$ (c) $x = \frac{e^y - e^{-y}}{2}$

$$e^y = e^{\ln(x + \sqrt{x^2-1})}$$

$$e^y = x + \sqrt{x^2-1}$$

$$(e^y - x)^2 = (\sqrt{x^2-1})^2$$

$$e^{2y} - 2xe^y + \cancel{x^2} = \cancel{x^2} - 1$$

$$e^{2y} + 1 = 2xe^y$$

$$\boxed{\frac{e^y + e^{-y}}{2} = \frac{(e^{-y})}{(e^{-y})}\frac{e^{2y}+1}{2e^y} = x}$$

Q: Find the length of curve $y = f(x) = \ln(x + \sqrt{x^2-1})$ on $[1, \sqrt{2}]$

$$L = \int_1^{\sqrt{2}} \sqrt{1 + \{f'(x)\}^2}\,dx$$

$\Rightarrow$? $\Leftarrow$ $\uparrow$ undefined @ $x = 1$

Write as an integral in y

when $x = 1$, $y = \ln(1+0)$, $y = 0$

when $x = \sqrt{2}$, $y = \ln(\sqrt{2}+1) = \ln(1+\sqrt{2})$

$$x = \frac{e^y + e^{-y}}{2} = g(y)$$

$$L = \int_0^{\ln(1+\sqrt{2})} \sqrt{1 + \{g'(y)\}^2}\,dy$$

I. Density and Mass:

Object with uniform density

[illegible] volume

Object with uniform density

mass = density · volume

Def: Suppose a thin bar is represented by interval $a \le x \le b$ with density function ρ (units mass/length). Then the mass of the object:

$$m = \int_a^b \rho(x)\,dx$$

Ex: A 2 meter bar is made of an alloy with density $\rho(x) = 1 + x^2$. What is the mass?

$$m = \int_0^2 (1+x^2)\,dx = \left[x + \frac{x^3}{3}\right]_0^2 = 2 + \frac{2^3}{3} = \boxed{\frac{14}{3}}$$

II. Work:

Work = Force × distance.

If F is constant, then $W = Fd$

If $F(x)$ is variable, then

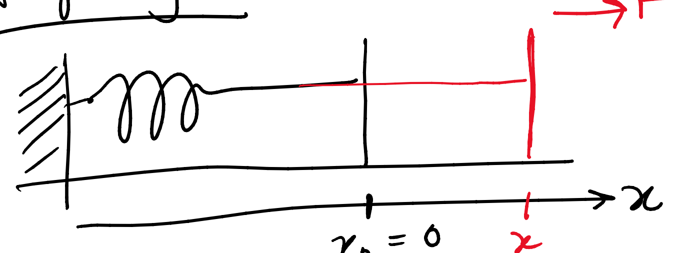
$$W = \int_a^b F(x)\,dx$$

the object is moving along a line from a to b

☆ Springs:

stretch

→ F



spring pointing in x direction

When the spring is at rest, it equilibrium position is $x = x_0$ (sometimes $x_0 = 0$)

Stretch the spring

stretch a spring

Stretch the spring

F - the force needed to stretch a spring x units away from equilibrium

Hooke's Law: The force F required to keep a spring in a stretched position x units away from the equilibrium position

$$F(x) = kx$$

k - spring constant - "stiffness"

x - displacement from equilibrium

Ex: A force of 10N is required to stretch a spring 0.1 m from equilibrium

(a) Find the spring constant K

Hooke's law

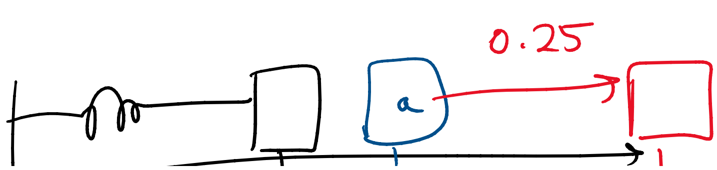
$$F = kx$$

$$10N = k(0.1m)$$

$$k = \frac{10N}{0.1m} = 100\frac{N}{m}$$

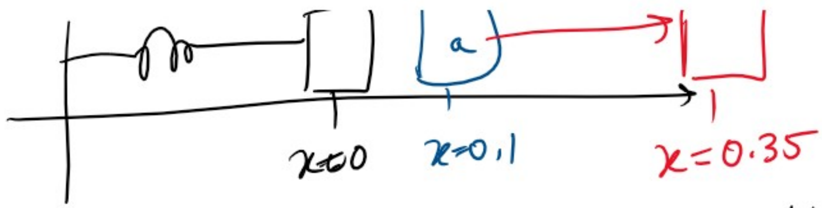
(c) How much work to stretch another 0.25m if it has been stretched 0.1 m?

$$W = \int_a^b F(x)\,dx = \int_{0.1}^{0.35} 100x\,dx$$



$$= \cdots$$

$$= \boxed{5.625 J}$$



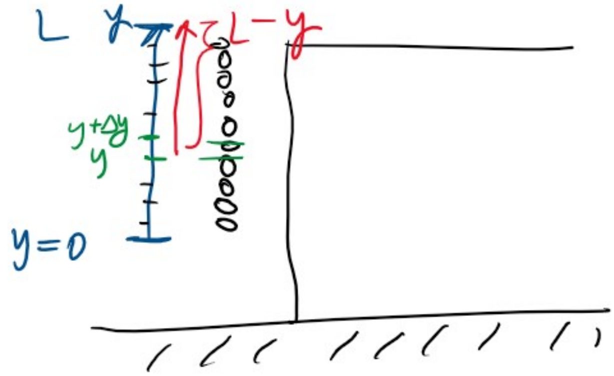
$= \boxed{5.625\,J}$

units of work → joules

III. Lifting Problems:

Lift an object → Force gravity

$W = \text{force} \cdot \text{distance} = (mg)\,y$



pulling a chain up

L – length of chain

rho → ρ – density (constant)

Δy – one segment of chain

@ y, work to lift one segment of chain?

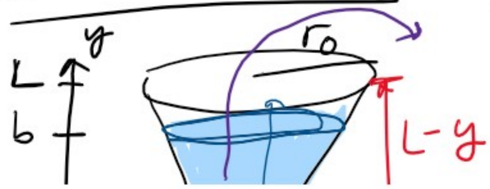
$$W_k = m \cdot g \cdot h = (\rho \Delta y)(g)(L-y)$$

$$W \approx \sum_{k=1}^{n} W_k \xrightarrow[\text{as } n\to\infty]{\text{limit}}$$

$$\boxed{W = \int_0^L \rho g (L-y)\,dy}$$

Lifting problems

IV. Pumping:



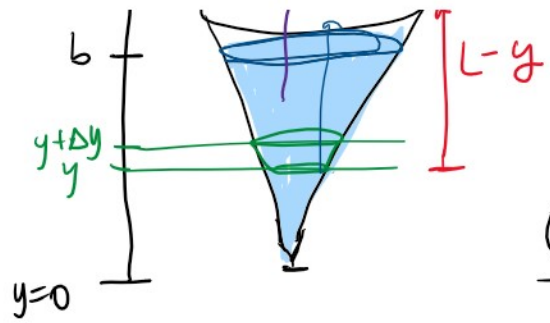
tank of water

r_0 – top radius of tank

b – height of water

ρ – density of water

L – height of tank



b - height of water $\qquad$ ρ - density of water

L - height of tank

Q: How much work to remove water?
(assuming its pumped from the top of the tank)

Work to remove one slice of water

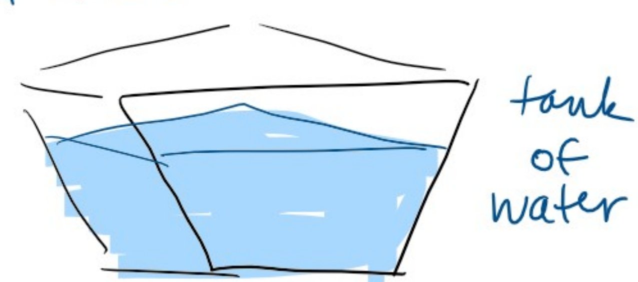
$$F_k = m \cdot g = \underbrace{A(y_k)\Delta y}_{\text{volume}} \underbrace{\rho}_{\text{density}} \cdot g$$

$$W_k = F_k \cdot d_k = A(y_k)\Delta y \rho g (L - y_k)$$

$$W \approx \sum_k W_k \xrightarrow{\text{limit}} \text{as } n \to \infty$$

$$\boxed{W = \int_0^b \rho g \underbrace{A(y)}_{\text{cross-section of tank}} (L-y)\, dy}$$

V. Force and Pressure:



$$\text{pressure} = \frac{\text{force}}{\text{area}}$$

Q: What is the pressure of the water on the sides of the tank?

Def: hydrostatic pressure is the pressure of water at rest.

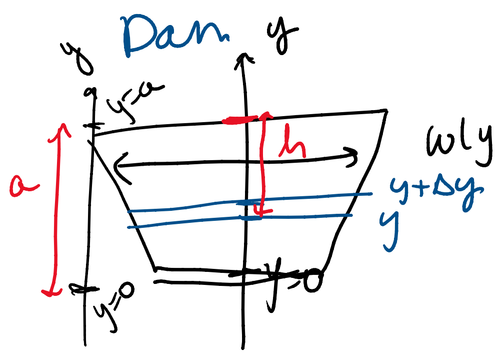
water at rest.

NOTE: hydrostratic pressure has the same magnitude in all directions

$$\text{pressure} = \frac{\text{force}}{\text{area}} = \rho g h$$

ρ - density of water

h - height



$w(y)$ — width of the face

$$F_k = \text{pressure} \cdot \text{area}$$

$$= \underbrace{\rho g \underbrace{(a-y)}_{\text{height}}}_{\text{pressure}} \underbrace{(w(y)\, \Delta y)}_{\text{area}}$$

$h = a - y$

a - height of dam

$$\sum F_k \xrightarrow{n \to \infty}$$

$$\boxed{F = \int_0^a \rho g (a-y)\, w(y)\, dy}$$

Force on a dam