6.7 Physical Applications

Announcements:
- exam 1 on Wed Feb 9 @ 6:30pm
- any one can attend in-person lectures now

GOALS:
- Find mass of thin bars
- Solve applications involving work
- Solve applications involving pressure & hydrostatic force

WARM UP: Express the function $y = f(x) = \ln(x + \sqrt{x^2-1})$ as a function in y, i.e. $x = g(y)$

$\boxed{(a)\ x = \frac{e^y + e^{-y}}{2}}$ (b) $x = \frac{e^{2y}+1}{2}$ (c) $x = \frac{e^y - e^{-y}}{2}$

$$e^y = e^{\ln(x+\sqrt{x^2-1})}$$

$$e^y = x + \sqrt{x^2-1}$$

$$(e^y - x)^2 = (\sqrt{x^2-1})^2$$

$$e^{2y} - 2xe^y + \cancel{x^2} = \cancel{x^2} - 1$$

$$e^{2y} + 1 = 2xe^y$$

$$x = \frac{e^{2y}+1}{2e^y}\left(\frac{e^{-y}}{e^{-y}}\right) = \frac{e^y + e^{-y}}{2} = \cosh(y)$$

$$\sinh(y) = \frac{e^y - e^{-y}}{2}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

Q: Find the length of curve $y = \ln(x + \sqrt{x^2-1})$ over $[1, \sqrt{2}]$

$$L = \int_1^{\sqrt{2}} \sqrt{1 + \left\{\frac{dy}{dx}\right\}^2}\, dx$$

integrand is undefined @ $x=1$

→ write this as integral in y

$$L = \int_0^{\ln(1+\sqrt{2})} \sqrt{1 + \{g'(y)\}^2}\, dy$$

$$x = \frac{e^y + e^{-y}}{2} = g(y)$$

@ $x = 1$
$y = \ln(1+0) = 0$
@ $x = \sqrt{2}$
$y = \ln(\sqrt{2}+1)$
$= \ln(1+\sqrt{2})$

I. Density and Mass:

rho

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uniform density (ρ) rho

$$\text{mass} = \text{density} \cdot \text{volume}$$

Thin bars or wire (essentially a 1D object)
on $[a,b]$

density function $\rho(x)$

then mass $\boxed{m = \int_a^b \rho(x)\,dx}$ mass of a thin bar

Ex: 2 m bar, density $\rho = 1 + x^2$

$$m = \int_0^2 (1+x^2)\,dx \longrightarrow m = \frac{14}{3}$$

II. Work:

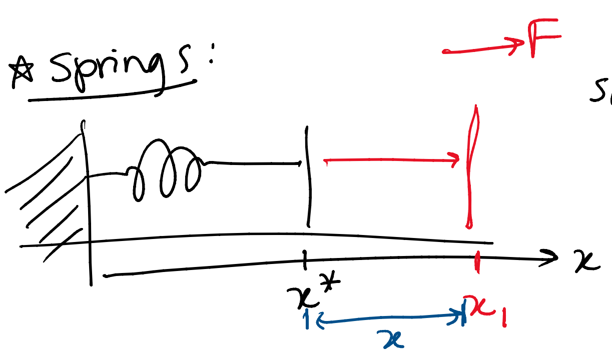
Work = Force × distance

F is constant $W = F \cdot d$

If $F(x)$ is variable, ← the object is moving along a line from a to b

$$W = \int_a^b F(x)\,dx$$

☆ Springs:



spring pointing in x-dir

when spring at rest ($F = 0$)

x^* - equilibrium position

Hooke's law: The force F required to keep a spring in a stretched position x-units from equilibrium is

$$F(x) = kx$$

k - spring constant

x - displacement from equilibrium

Ex: A force of 10N is required to stretch

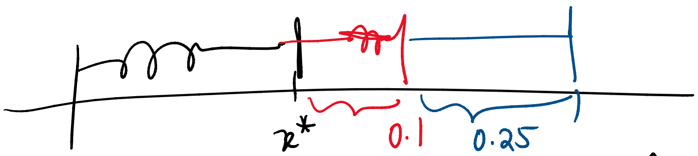
Ex: A force of 10N is required to stretch a spring 0.1 m from equilibrium.

(a) What is k?

$$F = kx$$
$$10N = k(0.1m)$$
$$k = \frac{10N}{0.1m} = 100\frac{N}{m}$$

(b) How much work to stretch another 0.25m if it is already stretched 0.1m?

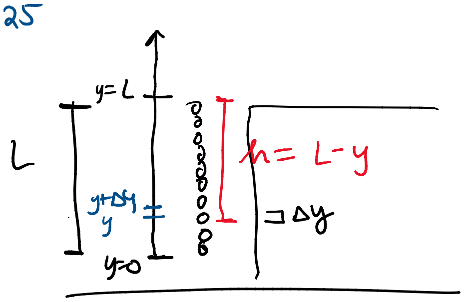
$$W = \int_a^b F(x)\,dx = \int_{0.1}^{0.35} 100x\,dx = \dots$$
$$= 5.625 J$$



J - joules
units of work

III. Lifting Problems:

Force is gravity



Lift a length of chain
- L - length of chain
- ρ - density
- Δy - one segment
- g - gravity constant

@y, the work to lift one segment of chain

$$W_k = m \cdot g \cdot \underbrace{h}_{\text{height to be lifted}} = \underbrace{(\rho\,\Delta y)}_{\text{mass}}\, g\, \underbrace{(L-y)}_{\text{height}}$$

$$W \approx \sum_{k=1}^{n} W_k \xrightarrow{\text{limit as } n\to\infty}$$

$$\boxed{W = \int_0^L \rho g (L-y)\,dy}$$

Lifting Problems

IV. Pumping:

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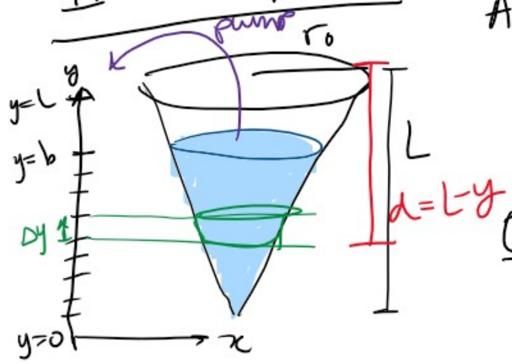


A tank of water is shaped like a cone

ρ – density of water

Q: How much work to empty the tank (must be pumped from the top)

b – height of water
L – height of tank
$y=0$. bottom of tank

$A(y)$ – cross-section of one slice

Force on one slice?

$$F_k = mg = \underbrace{A(y)\Delta y}_{\text{volume}} \underbrace{\rho}_{\text{density}} \cdot g$$

Work on one slice

$$W_k = F_k \cdot d_k = A(y)\Delta y \rho g (L - y)$$

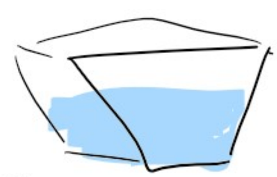
$$W \approx \sum_{k=1}^{n} W_k \longrightarrow n \to \infty$$

$$\boxed{W = \int_0^b \rho g \, A(y)(L-y)\, dy}$$

pumping out of a tank.

V. Force and Pressure:

tank of water



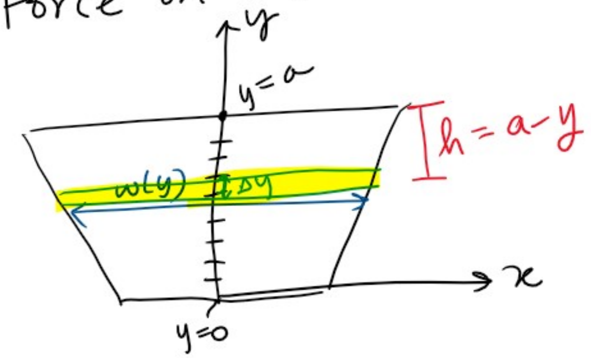
Q: What is the pressure of the water on the walls of the tank?

Def: hydrostatic pressure is the of water at rest
↳ same magnitude in all directions

$$\text{pressure} = \frac{\text{force}}{\text{area}} = \rho g h$$

ρ – density of water
h – height

Force on a dam



ρ – density of water
a – height of dam
w(y) – width of the dam

Force on one segment

$$F_k = \text{pressure} \cdot \text{area}$$
$$= \rho g h \cdot (w(y)\Delta y)$$
$$= \rho g (a-y)\, w(y) \Delta y$$

$$F \approx \sum_{k=1}^{n} F_k \xrightarrow{\quad n \to \infty \quad}$$

$$\boxed{F = \int_0^a \rho g (a-y)\, w(y)\, dy}$$

Force on a dam