Exam 2 Review

Continue Summary of material

→ Then problems at end

Lesson 16-17: Partial fractions

$$\int \frac{P(x)}{Q(x)}\,dx \qquad \text{where degree } P(x) < \text{degree } Q(x)$$

type	eqn	decomposition
simple linear	$\frac{P(x)}{(x-1)(x-2)(x-3)}$	$\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$
repeated linear	$\frac{P(x)}{(x-1)^3}$	$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$
simple quadratic	$\frac{P(x)}{x^2+x+1}$	$\frac{Ax+B}{x^2+x+1}$
repeated quadratic	$\frac{P(x)}{[x^2+x+1]^2}$	$\frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{[x^2+x+1]^2}$

then $\frac{P(x)}{Q(x)}$ = partial fraction decomposition

Multiply both sides by common denominator

Solve for constants $A, B, C, \ldots$

2 methods:

1. Collect like terms
2. Evaluate at points x

zeros of common denominator

② Evaluate at points x

try: - roots of common denominator

- simple values like $x = 0, 1, 2 \ldots$

Warning: $\int \frac{P(x)}{Q(x)}\,dx$

If degree $P(x) \geq$ degree of $Q(x)$

Need to use polynomial division first

$$Q(x)\,\overline{\big)\; P(x)\;}^{\;F(x)}$$

with remainder $R(x)$

then $\frac{P(x)}{Q(x)} = F(x) + \frac{R(x)}{Q(x)}$ ← now degree $R(x) <$ degree $Q(x)$

(will do an example below)

Lesson 18: Improper Integrals

$$\int_1^\infty f(x)\,dx = \lim_{b\to\infty} \int_1^b f(x)\,dx$$

Q: What if $f(x)$ has a point that is undefined on $[a,b]$?

$\boxed{\text{case 1}}$ $f(x)$ undefined @ $x = a$

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$

$\boxed{\text{case 2}}$ $f(x)$ undefined @ $x = b$

$\boxed{\text{Case 2}}$ $f(x)$ undefined @ $x = b$

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

$\boxed{\text{case 3}}$ $f(x)$ undefined @ $x = z$, where $a < z < b$

$$\int_a^b f(x)\,dx = \lim_{c \to z^-} \int_a^c f(x)\,dx + \lim_{c \to z^+} \int_c^b f(x)\,dx$$

<u>Comparison Test</u>: $\int_a^b f(x)\,dx$

Two ways to use

① If $0 < f(x) \le g(x)$

Then $0 < \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

If this converges then $\int_a^b f(x)\,dx$ converges

② If $h(x) \le f(x)$

Then $\int_a^b h(x)\,dx \le \int_a^b f(x)\,dx$

If this <u>diverges</u> then $\int_a^b f(x)\,dx$ diverges

<u>Past Exam Problems</u>:

Fall 2018 Exam 2 #7

Past Exam Problems:

Fall 2019 Exam 2 # 7

7. Evaluate $\int_0^2 \frac{x^2}{x^2+2x+2}\,dx$. There are several ways to do this; the general method of partial fractions is among the quickest.

A. 3π

B. $1+\pi/2$

C. $2-\ln 5$

D. $\ln 6$

E. $64/15$

$$\int_0^2 \frac{x^2}{x^2+2x+2}\,dx$$

degree 2

also degree 2

Need to do polynomial division first

$\boxed{1}$ F(x)

$$\begin{array}{r|l} x^2+2x+2 & x^2 \\ & -(x^2+2x+2) \\ \hline & 0 \ \boxed{-2x-2} \end{array}$$

Remainder R(x)

$$\int_0^2 \frac{x^2}{x^2+2x+2}\,dx = \int_0^2 1 + \frac{-2x-2}{x^2+2x+2}\,dx$$

use u-sub

$u = x^2+2x+2$

$du = (2x+2)\,dx$

$$= \int_0^2 dx + \int_2^{10} -\frac{du}{u}$$

$$= \left[x\right]_0^2 + \left[-\ln u\right]_2^{10}$$

$$= \left[2-0\right] + \left[-\ln 10 + \ln 2\right]$$

$$= 2 - \ln\left(\frac{10}{2}\right) = \boxed{2-\ln(5)} \quad \boxed{C}$$

★ Fall 2019 Exam 2 #11

11. The integral $\int_1^\infty \frac{e^{-t}+1}{t}\,dt$

A. is convergent, by comparison with $\int_1^\infty \frac{1}{t}\,dt$;

B. is divergent, by comparison with $\int_1^\infty \frac{1}{t}\,dt$;

C. is convergent, by comparison with $\int_1^\infty e^{-t}\,dt$;

D. is divergent, by comparison with $\int_1^\infty e^{-t}\,dt$.

E. None of the above statements is correct.

Want to use comparison test

We know:

$$0 < \frac{1}{t} \leq 1 \quad \text{on } [1,\infty)$$

$$0 < e^{-t} \leq 1 \quad \text{on } [1,\infty)$$

$$1 < e^{-t}+1 \leq 2 \quad \text{on } [1,\infty)$$

Comparison Test

$$\frac{1}{t} < \frac{e^{-t}+1}{t}$$

so $$\int_1^\infty \frac{1}{t}\,dt < \int_1^\infty \frac{e^{-t}+1}{t}\,dt$$

$$\lim_{b\to\infty}\int_1^b \frac{1}{t}\,dt = \lim_{b\to\infty}\Big[\ln t\Big]_1^b$$

$$= \lim_{b\to\infty}\left[\ln b - \ln 1\right] = +\infty$$

so $$+\infty < \int_1^\infty \frac{e^{-t}+1}{t}\,dt$$

The integral diverges, by comparison

The integral diverges, by comparison with $\int_1^\infty \frac{1}{t}\,dt$ $\boxed{B}$

☆ Trig subst problem: 8.4 #11

$$\int_{1/2}^{\sqrt{3}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$x = \sin\theta$

$dx = \cos\theta\,d\theta$

$\frac{\sqrt{3}}{2} = \sin\theta \rightarrow \theta = \frac{\pi}{3}$

$\frac{1}{2} = \sin\theta \rightarrow \theta = \frac{\pi}{6}$

$$= \int_{\frac{\pi}{6}}^{\pi/3} \frac{\sin^2\theta \cdot \cos\theta\,d\theta}{\sqrt{1-\sin^2\theta}} = \int_{\frac{\pi}{6}}^{\pi/3} \frac{\sin^2\theta\,\cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}}$$

$$= \int_{\frac{\pi}{6}}^{\pi/3} \sin^2\theta\,d\theta$$

Half-angle formula

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\cos(2\theta)}{2} = \frac{1}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left[\frac{\pi}{3} - \frac{\sin\left(\frac{2\pi}{3}\right)}{2} - \frac{\pi}{6} + \frac{\sin\left(\frac{2\pi}{6}\right)}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - \cancel{\frac{\sqrt{3}}{2\cdot 2}} + \cancel{\frac{\sqrt{3}}{2\cdot 2}}\right] = \frac{\pi}{12}$$

☆ Spring 2020 Exam 2 #11

11. Compute the integral $\int \frac{x}{\sqrt{5+4x-x^2}}dx.$

A. $2\cos^{-1}\left(\frac{x-2}{3}\right) - \sqrt{5+4x-x^2} + C$

complete the square first

$5 + \cancel{4x} - \cancel{x^2} = -(x-2)^2 + 6$

$= -(x^2 - 4x + 4) +$

A. $2\cos^{-1}\left(\frac{x-2}{3}\right) - \sqrt{5+4x-x^2} + C$

B. $2\sin^{-1}\left(\frac{x-2}{3}\right) - \sqrt{5+4x-x^2} + C$

C. $2\sin^{-1}\left(\frac{x-2}{3}\right) + \sqrt{5+4x-x^2} + C$

D. $2\cos^{-1}\left(\frac{\sqrt{5+4x-x^2}}{3}\right) - \sqrt{5+4x-x^2} + C$

E. $2\cos^{-1}\left(\frac{x-2}{3}\right) - \frac{\sqrt{5+4x-x^2}}{2} + C$

$$5 + \cancel{4x} - \cancel{x^2} = -(x-2)^2 + b$$
$$= -(x^2-4x+4)+b$$
$$= \cancel{-x^2} + \cancel{4x} - 4 + b$$

$$5+4 = b$$
$$b = 9$$

$$\int \frac{x}{\sqrt{5+4x-x^2}}\,dx = \int \frac{x}{\sqrt{9-(x-2)^2}}$$

u-sub

$u = x-2$

$du = dx$

$x = u+2$

$$= \int \frac{u+2}{\sqrt{9-u^2}}\,du$$

$$= \int \frac{u}{\sqrt{9-u^2}}\,du + \int \frac{2\,du}{\sqrt{9-u^2}}$$

another subst.

$v = 9-u^2$

$dv = -2u\,du$

trig subst.

$u = 3\sin\theta$

$du = 3\cos\theta\,d\theta$

$$= \frac{-1}{2}\int \frac{dv}{v^{1/2}} + 2\int \frac{3\cos\theta\,d\theta}{\sqrt{3^2(1-\sin^2\theta)}}$$

$$= \frac{-1}{2}\int v^{-1/2}\,dv + 2\int \frac{\cancel{3\cos\theta}\,d\theta}{\cancel{3\cos\theta}}$$

$$= \frac{-1}{2}\left[\frac{v^{1/2}}{1/2}\right] + 2\int d\theta$$

$$= -\sqrt{v} + 2\theta + C$$

$v = 9-u^2$

want answer in terms of x

$u = 3\sin\theta$

$\sin\theta = \frac{u}{}$

Want answer in terms of x

$u = 3\sin\theta$
$\sin\theta = \frac{u}{3}$
$\theta = \sin^{-1}\left(\frac{u}{3}\right)$
$u = x-2$

$$= -\sqrt{9-u^2} + 2\sin^{-1}\left(\frac{u}{3}\right) + C$$

$$= -\sqrt{9-(x-2)^2} + 2\sin^{-1}\left(\frac{x-2}{3}\right) + C$$

$$= \boxed{2\sin^{-1}\left(\frac{x-2}{3}\right) - \sqrt{5+4x-x^2} + C}$$

$\boxed{B}$

☆ Fall 2019 Exam 2 #8

8. $\int_1^3 x^2 \ln\frac{x}{3}\,dx =$

A. $\frac{\ln 3}{3} - \frac{26}{9}$

B. $-9 + \frac{\ln 3}{15}$

C. 0

D. $3\ln 6 + \frac{3}{5}$

E. $12 - \frac{3\ln 3}{16}$

← first, use u-substitution to get rid of the $\frac{x}{3}$ in the ln

Call new variable y

Let $y = \frac{x}{3}$ $\quad dy = \frac{dx}{3}$

$x = 3y$ $\quad dx = 3dy$

$$\int_1^3 x^2 \ln\left(\frac{x}{3}\right)dx = \int_{\frac{1}{3}}^1 (3y)^2 \ln(y)(3dy)$$

$$= 27\int_{\frac{1}{3}}^1 y^2 \ln(y)\,dy$$

Use integration by Parts

$u = \ln(y)$ $\quad dv = y^2 dy$

$du = \frac{1}{y}dy$ $\quad v = \frac{y^3}{3}$

$$= 27\left[u \cdot v - \int v\,du\right]$$

$$= 27\left[\ (\ln)\ y^3 \quad \int^1 y^3 \quad 1 \quad du\right]$$

$$= 27\left[\ln(y)\cdot\frac{y^3}{3} - \int_{\frac{1}{3}}^{1}\frac{y^3}{3}\cdot\frac{1}{y}\,dy\right]$$

$$= \frac{27}{3}y^3\ln(y) - \frac{27}{3}\int y^2\,dy$$

$$= \left[9y^3\ln(y) - 9\,\frac{y^3}{3}\right]_{\frac{1}{3}}^{1}$$

$$= \left[9y^3\ln(y) - 3y^3\right]_{\frac{1}{3}}^{1}$$

$$= \left[9\cdot 1^3\,\cancel{\ln(1)}^{\,0} - 3(1)^3 - 9\cdot\left(\frac{1}{3}\right)^3\ln\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right)^3\right]$$

$$= -3 - \frac{9}{27}\ln\left(\frac{1}{3}\right) + \frac{3}{27}$$

$\ln\left(\frac{1}{3}\right) = -\ln(3)$

$$= -\frac{1}{3}\left[-\ln(3)\right] - 3 + \frac{1}{9}$$

$$= \frac{\ln(3)}{3} + \left[\frac{-3\cdot 9 + 1}{9}\right] = \boxed{\frac{\ln(3)}{3} - \frac{26}{9}}\quad \boxed{A}$$