Exam 3 Review - Friday - LEC 200

Warm Up: Which of the following series converge?

I. $\displaystyle\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$ DIV

II. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}\sin\left(\frac{1}{k}\right)$ CONV

~~(a) I only~~

(b) II only

~~(c) both converge~~

(d) neither converge

I. L.C.T.

$$L = \lim_{k\to\infty} \frac{\sin\left(\frac{1}{k}\right)}{\frac{1}{k}} \overset{L'H}{=} \lim_{k\to\infty} \frac{\cos\left(\frac{1}{k}\right)\left(\cancel{\frac{-1}{k^2}}\right)}{\cancel{\frac{-1}{k^2}}} = 1$$

$L=1$ $\quad \sum \frac{1}{k}$ diverges $\Rightarrow \sum \sin\left(\frac{1}{k}\right)$ diverges

II. L.C.T

$$L = \lim_{k\to\infty} \frac{\cancel{\frac{1}{k}}\sin\left(\frac{1}{k}\right)}{\frac{1}{k^{\cancel{2}}}} = \lim_{k\to\infty} \frac{\sin\left(\frac{1}{k}\right)}{\frac{1}{k}} = 1$$

$L=1$ $\quad \sum \frac{1}{k^2}$ p series converges $\rightarrow \sum \frac{1}{k}\sin\left(\frac{1}{k}\right)$ converges

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S19 #10: The power series representation of the function $\frac{x}{1+x^2}$ is given by:

$$\frac{x}{1+x^2} = x\left(\frac{1}{1+x^2}\right) = x\left(\frac{1}{1-(-x^2)}\right) \leftarrow \text{Geometric series}$$

$$\infty \qquad \left(\sum^{\infty} \ldots x^{k} \ldots {}^{2k}\right)$$

$$\frac{x}{1+x^2} = x\left(\frac{1}{1+x^2}\right) = x\left(\frac{1}{1-(-x^2)}\right)$$

$$= x\left(\sum_{k=0}^{\infty}(-x^2)^k\right) = x\left(\sum_{k=0}^{\infty}(-1)^k x^{2k}\right)$$

$$= \sum_{k=0}^{\infty}(-1)^k x^{2k+1}$$

all of the possible answers start with $n=1$

$$\boxed{= \sum_{n=1}^{\infty}(-1)^{n-1} x^{2n-1}}$$

Let $n = k+1$

$n-1 = k$

$2k+1 = 2(n-1)+1 = 2n-2+1$

$= 2n-1$

$$\sum_{k=0}^{\infty}(-1)^k x^{2k+1} = (-1)^0 x^1 + \sum_{k=1}^{\infty}(-1)^k x^{2k+1}$$

$$= x + \sum_{k=1}^{\infty}(-1)^k x^{2k+1}$$

Alternating Series Estimation Theorem:

Given an alternating series $\sum_{k=0}^{\infty}(-1)^{k+1} a_k = S$

where $a_k \geq 0$

Let $S_n = \sum_{k=0}^{n}(-1)^{k+1} a_k$ is the partial sum

Then

$$\boxed{|S - S_n| = |R_n| \leq a_{n+1}}$$

Ex: F18 #7

Let $S = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1}{m^4}$ and S_n is the partial sum

According to the Thm. what is the smallest n such that $|S - S_n| < 4^4 \times 10^{-8}$

$$|S - S_n| \le a_{n+1} = \frac{1}{(n+1)^4} < 4^4 \times 10^{-8} = \frac{4^4}{10^8}$$

$$\sqrt[4]{\frac{10^8}{4^4}} < \sqrt[4]{(n+1)^4}$$

$$25 = \frac{100}{4} = \frac{10^2}{4} < n+1$$

$$24 < n$$

Need $\boxed{n = 25}$

Taylor Series Review:

Taylor series of $f(x)$ centered at a is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Maclaurin series of $f(x)$ is the Taylor series centered at $a = 0$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Taylor polynomial $p_n(x)$ is a partial sum

$$\sum^{n} f^{(k)}(a) \, (x-a)^k$$

Taylor polynomial $p_n(x)$

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Taylor Remainder Est. Thm:

$$|R_n(x_0)| = |f(x_0) - p_n(x_0)| \le \frac{M\,|x_0 - a|^{n+1}}{(n+1)!}$$

$$\text{where } M = \max_{a \le x \le x_0} \left| f^{(n+1)}(x) \right|$$

F19 - Exam 3 #10

Let $p_3(x)$ be the Taylor poly, centered at 0, of e^{-x}. Remainder Thm guarantees that $p_3(1)$ approximates e^{-1} with error $\le$?

$$|R_3(1)| \le \frac{M\,|1-0|^{3+1}}{(3+1)!}$$

$$M = \max_{0 \le x \le 1} \left| f^{(4)}(x) \right| = \max_{0 \le x \le 1} \left| \frac{d^4}{dx^4} e^{-x} \right|$$

$$= \max_{0 \le x \le 1} \left| \pm e^{-x} \right| = e^{-0} = 1$$

$$|R_3(1)| \le \frac{1 \cdot 1^4}{4!} = \boxed{\frac{1}{24}}$$

HOTSEAT: let $p_2(x)$ be the 2nd order Taylor poly of $f(x) = \ln(x)$. The

centered at $a=1$ ...

HOTSEAT: Let $p_2(x)$ be the
centered at $a=1$ of $f(x)=\ln(x)$. The
estimate $p_2(2) \approx \ln(2)$ has error $|R_2(2)| \le ?$

(a) $\frac{1}{3}$

(b) $\frac{16}{3}$

(c) $\frac{8}{3}$

(d) $\frac{1}{12}$

$$|R_n(x_0)| \le \frac{M|x_0-a|^{n+1}}{(n+1)!}$$

$$|R_2(2)| \le \frac{M|2-1|^3}{3!} = \frac{M}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

$$M = \max_{1\le x\le 2} |f'''(x)| = \max_{1\le x\le 2} \left|\frac{2}{x^3}\right| = 2$$

$$M = \max_{a\le x\le x_0} |f^{(n+1)}(x)|$$

S16 #9:

Let $\sin(x) = \sum_{n=0}^{\infty} a_n \left(x-\frac{\pi}{6}\right)^n$ be the T.S. for $\sin(x)$ at $a=\frac{\pi}{6}$. Then $a_2 = ?$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(\frac{\pi}{6}\right)}{n!}\left(x-\frac{\pi}{6}\right)^n$$

$$a_n = \frac{f^{(n)}\left(\frac{\pi}{6}\right)}{n!} \qquad n=2$$

$$a_2 = \frac{-\sin\left(\frac{\pi}{6}\right)}{2} = \frac{-\left(\frac{1}{2}\right)}{2} = \boxed{-\frac{1}{4}}$$

NOTE: "What is the coefficient of the $\left(x-\frac{\pi}{6}\right)^2$

NOTE: "What is the coefficient of the $(x-\frac{\pi}{6})^2$ term?"

Asks the same question $\to a_2 = ?$

S19 #11: Given that $\arctan(x^2) = \int_0^x \frac{2t}{1+t^4}\,dt$

is the Maclaurin series of the fcn $\arctan(x^2)$ for $|x|<1$ is given by: ?

$$\arctan(x^2) = \int_0^x \frac{2t}{1+t^4}\,dt = \int_0^x 2t\left(\frac{1}{1-(-t^4)}\right)dt$$

Geometric series

$$= \int_0^x 2t\left(\sum_{k=0}^{\infty}(-t^4)^k\right)dt$$

$$= \int_0^x 2t \sum_{k=0}^{\infty}(-1)^k t^{4k}\,dt$$

$$= \int_0^x \sum_{k=0}^{\infty} 2(-1)^k t^{4k+1}\,dt$$

$$= \sum_{k=0}^{\infty} 2(-1)^k \int_0^x t^{4k+1}\,dt$$

$$= \sum_{k=0}^{\infty} 2(-1)^k \left[\frac{t^{4k+2}}{4k+2}\right]_0^x$$

$$\sum^{\infty} \quad (-1)^k \; t^{4k+2}$$

$$\sum^{\infty} (-1)^k t^{4k+2}$$

$$= \sum_{k=0}^{\infty} \cancel{2}(-1)^k \frac{t^{4k+2}}{\cancel{2}(\cancel{4}k+\cancel{2})} = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k t^{4k+2}}{2k+1}}$$

Differentiating

Find the power series $\frac{-1}{(1+x)^2}$

Note:

$$\frac{-1}{(1+x)^2} = \frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{d}{dx}\left(\frac{1}{1-(-x)}\right)$$

Geometric series

$$= \frac{d}{dx} \sum_{k=0}^{\infty} (-x)^k = \frac{d}{dx} \sum_{k=0}^{\infty} (-1)^k x^k$$

$$= \frac{d}{dx}\left(\cancel{(-1)^0 x^0}\right) \to 0 + \frac{d}{dx} \sum_{k=1}^{\infty} (-1)^k x^k$$

$$= \sum_{k=1}^{\infty} \frac{d}{dx}\left[(-1)^k x^k\right] = \boxed{\sum_{k=1}^{\infty} (-1)^k k x^{k-1}}$$