Exam 3 Review - Monday - LEC 286

In Hot Seat Thoughts, you can request topics/problems to review on Wednesday

Announcements:

New calendar dates:

- HW 33 due Mon 4/25
- HW 34 due Wed 4/27

Office Hours tomorrow Tues @ 1-2pm
Zoom link on LEC Brightspace

☆ Warm Up:

Is the following statement true or false?

If a sequence a_n is such that $-n-1 \le \ln|a_n| \le -n$ then $\sum_{n=1}^{\infty} a_n$ converges (S19 - Exam 3 - #7 - Part III)

(Hint: use the Root Test)

(a) True (b) False

Root Test: $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{1/n}$

$$= e^{\lim_{n\to\infty} \ln\left(|a_n|^{1/n}\right)} = e^{\lim_{n\to\infty} \frac{1}{n}\ln|a_n|}$$

$$-1-\frac{1}{n} = \frac{-n-1}{n} \le \frac{\ln|a_n|}{n} \le \frac{-n}{n} = -1$$

$$\frac{\ln|a_n|}{n} \xrightarrow{n\to\infty} -1$$

$$\rho = e^{-1} = \frac{1}{e} < 1 \rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

Exam 3 Review

multiplication

NOTE: We did not cover multiplication of power series. In Past Exams, you can skip:

$e^{-2x}\sin(3x)$ F15 #11

$\sin(x)\cos(x)$ F16 #11

$\tan(x) = \frac{\sin(x)}{\cos(x)}$ F16 #12

Computing value of series

S13 - Exam 3 - #1:

Compute $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n-1} - 3^{n+1}}{4^n}$

separate $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n-1}}{4^n} - \sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n}$

Geometric Series $\displaystyle\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ if $|r|<1$

let $k = n-1$
$k+1 = n$

$$\sum_{k=0}^{\infty} \frac{2^k}{4^{k+1}} - \sum_{k=0}^{\infty} \frac{3^{k+2}}{4^{k+1}}$$

$$= \sum_{k=0}^{\infty} \underbrace{\frac{1}{4}}_{a} \underbrace{\left(\frac{2}{4}\right)^k}_{r=\frac{2}{4}<1} - \sum_{k=0}^{\infty} \underbrace{\frac{3^2}{4}}_{a} \underbrace{\left(\frac{3}{4}\right)^k}_{r=\frac{3}{4}<1}$$

$$= \frac{\left(\frac{1}{4}\right)}{1-\left(\frac{1}{2}\right)} - \left(\frac{\left(\frac{9}{4}\right)}{1-\frac{3}{4}}\right) = \frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{2}\right)} - \frac{\left(\frac{9}{4}\right)}{\left(\frac{1}{4}\right)}$$

$$= \frac{2}{4} - 9 = \boxed{\frac{-17}{2}}$$

Integral of Power Series:

S18 - Exam 3 - #10:

Find the power series $f(x)$ for $\int \frac{x^2}{1+4x^2}\,dx$ centered at zero. Find the radius of convergence.

$$\int \frac{x^2}{1+4x^2}\,dx = \int x^2\left(\frac{1}{1+4x^2}\right)dx$$

Geometric series $\quad \frac{1}{1-r} = \sum_{k=0}^{\infty} r^k \quad |r|<1$

let $r = -4x^2$

$$\int x^2\left(\frac{1}{1-(-4x^2)}\right)dx = \int x^2\left(\sum_{k=0}^{\infty}(-4x^2)^k\right)dx$$

$$= \int x^2\left(\sum_{k=0}^{\infty}(-1)^k 4^k x^{2k}\right)dx$$

$$= \int \sum_{k=0}^{\infty}(-1)^k 4^k x^{2k+2}\,dx$$

$$= \sum_{k=0}^{\infty}(-1)^k 4^k \int x^{2k+2}\,dx$$

$$= \sum_{k=0}^{\infty} (-1)^k 4^k \int x \quad dx$$

$$= \left(\sum_{k=0}^{\infty} (-1)^k 4^k \frac{x^{2k+3}}{2k+3} \right) + C$$

Radius of Convergence:

If $f(x)$ has R, then $\int f(x)\,dx$ has R

used Geometric series

$$|r| < 1$$

$$|-4x^2| < 1$$

$$4|x|^2 < 1$$

$$|x|^2 < \frac{1}{4}$$

$$\boxed{R = \frac{1}{2}} \longrightarrow |x| < \frac{1}{2}$$

Functions $\Longrightarrow$ Power Series

$$f(x) \longrightarrow \sum_{k=0}^{\infty} c_k (x-a)^k$$

use one of common Maclaurin series

know these

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad |x| < 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \left(\text{because } \frac{d^k}{dx^k}(e^x) = e^x \right)$$

$$\sum^{\infty} (-1)^k x^{2k+1}$$

These will be given

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

These will be given in the statement of problem

Ex: $\ln|1-x| = -\int \frac{dx}{1-x}$

$u = 1-x$
$du = -dx$
$= \int \frac{du}{u} = \ln|1-x|$

$$= -\int \sum_{k=0}^{\infty} x^k \, dx = -\sum_{k=0}^{\infty} \int x^k \, dx$$

$$= -\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} + C$$

Find C by plugging in $x = 0$

Power Series $\longrightarrow$ Functions

look for common maclaurin series

Ex: $\displaystyle\sum_{k=0}^{\infty} 2^k x^{2k+1} = \sum_{k=0}^{\infty} (2x^2)^k (x)$

$$= x\left(\sum_{k=0}^{\infty} \underbrace{(2x^2)}_{r}{}^k\right) = x\left(\frac{1}{1-2x^2}\right) = \frac{x}{1-2x^2}$$

If – Then Type Problems: Which of the following

If – Then Type Problems: Which of the following statements are true?

S19 - Exam 3 - #5

① If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges

→ looks like an alternating series
Need $a_n \geq 0$

Counterexample:

$a_n = \frac{(-1)^{n+1}}{n}$ Alt. Harm Series → converges

$(-1)^n a_n = \frac{(-1)^n (-1)^{n+1}}{n} = \frac{(-1)^{2n+1}}{n} = \frac{-1}{n}$ FALSE

$\sum (-1)^n a_n = -\left(\sum \frac{1}{n}\right)$ → diverges

② HOT SEAT
If $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} (-1)^n a_n$ converges

TRUE:

Comparison Test

$(-1)^n a_n \leq |a_n|$ → converges

By Comp. Test $\sum (-1)^n a_n$ converges.

③ $\lim_{n\to\infty} n^2 |a_n| = 1 \Rightarrow \sum_{n=1}^{\infty} (-1)^n a_n$ converges absolutely

Want $\sum |(-1)^n a_n|$ converges

$\sum |a_n|$ converges

Want $\sum (-1)^n a_n$ converges

$\sum |a_n|$ converges

L.C.T.

Given $L = \lim_{n\to\infty} n^2 |a_n| = \lim_{n\to\infty} \frac{|a_n|}{\frac{1}{n^2}} = 1$

$b_n = \frac{1}{n^2}$ p-series $p=2$ converges

L.C.T. $L=1$ and $\sum b_n$ converges

$\Rightarrow \sum |a_n|$ converges

$\Rightarrow \sum (-1)^n a_n$ converges

(4) If $0 \le b_n \le \frac{1}{\sqrt{n}}$ and $b_n \ge b_{n+1}$ for $n \ge 1$

$\Rightarrow \underbrace{\sum_{n=1}^{\infty} (-1)^n b_n}_{\text{Alternating series}}$ converges

$b_n \ge 0$

A.S.T: (a) nonincreasing ✓

(b) $\lim_{n\to\infty} b_n = 0$ ✓ $\rightarrow \sum (-1)^n b_n$ converges

<u>TRUE</u>

$0 \le b_n \le \frac{1}{\sqrt{n}}$

$\downarrow \quad \downarrow \; n \to \infty$

$0 \qquad 0$