Exam 3 Review - Friday - LEC 286

Warm Up: Which of the following series converge?

I. $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$ II. $\sum_{k=1}^{\infty} \frac{1}{k}\sin\left(\frac{1}{k}\right)$

~~(a) I only~~

$\boxed{\text{(b) II only}}$

~~(c) both converge~~

(d) neither converge

I. L.C.T.

$$L = \lim_{k\to\infty} \frac{\sin\left(\frac{1}{k}\right)}{\frac{1}{k}} \overset{L'H}{=} \lim_{k\to\infty} \frac{\cos\left(\frac{1}{k}\right)\cancel{\left(\frac{-1}{k^2}\right)}}{\cancel{\left(\frac{-1}{k^2}\right)}} = 1$$

$L = 1$ $\sum \frac{1}{k}$ diverges $\Rightarrow \sum \sin\left(\frac{1}{k}\right)$ diverges

II. L.C.T.

$$L = \lim_{k\to\infty} \frac{\frac{1}{\cancel{k}}\sin\left(\frac{1}{k}\right)}{\frac{1}{k^{\cancel{2}}}} = \lim_{k\to\infty} \frac{\sin\left(\frac{1}{k}\right)}{\frac{1}{k}} = 1$$

$L = 1$ $\sum \frac{1}{k^2}$ p-series $p=2$ converges $\Rightarrow \sum \frac{1}{k}\sin\left(\frac{1}{k}\right)$ converges

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Assigned Seats for Exam 3

LEC Brightspace $\to$ Gradebook

S19 #10: The power series representation of

$\frac{x}{1+x^2}$ is given by: ?

$\frac{x}{1+x^2}$ is given by..

$$\frac{x}{1+x^2} = x\left(\frac{1}{1+x^2}\right) = x\left(\frac{1}{1-(-x^2)}\right)$$

Geometric Series

$$= x\sum_{k=0}^{\infty}(-x^2)^k = x\sum_{k=0}^{\infty}(-1)^k x^{2k}$$

$$= \sum_{k=0}^{\infty}(-1)^k x^{2k+1}$$

All the possible answers starts with $n=1$

let $n = k+1$

$n-1 = k$

$2k+1 = 2(n-1)+1$

$= 2n-2+1 = 2n-1$

$$\boxed{= \sum_{n=1}^{\infty}(-1)^{n-1} x^{2n-1}}$$

Alternating Series Estimation Thm:

Given an alternating series $\sum_{k=0}^{\infty}(-1)^{k+1} a_k = S$

where $a_k \geq 0$

the partial sum $S_n = \sum_{k=0}^{n}(-1)^{k+1} a_k$

Then $\boxed{|S - S_n| = |R_n| \leq a_{n+1}}$

Ex: F18 #7

$S = \sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{m^4}$ and it's partial sum S_n

... is the smallest n

$S = \sum_{m=1} \ldots \frac{\ }{m^4}$

According to Thm, what is the smallest n such $|S - S_n| < 4^4 \times 10^{-8}$

$$|S - S_n| \leq a_{n+1} = \frac{1}{(n+1)^4} < 4^4 \times 10^{-8} = \frac{4^4}{10^8}$$

$$\sqrt[4]{\frac{10^8}{4^4}} < \sqrt[4]{(n+1)^4}$$

$$25 = \frac{100}{4} = \frac{10^2}{4} < n+1$$

$$24 < n$$

$$\boxed{n = 25}$$

Taylor Series Review:

Taylor series: of $f(x)$ centered at a is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Maclaurin series of $f(x)$ is the Taylor series when $a = 0$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

Taylor polynomial $p_n(x)$ is the partial sum of the Taylor series

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Taylor Remainder Estimation Theorem:

$$|R_n(x_0)| = |f(x_0) - p_n(x_0)| \le \frac{M\,|x_0 - a|^{n+1}}{(n+1)!}$$

Where $M = \max\limits_{a \le x \le x_0} |f^{(n+1)}(x)|$

FIG #10:

Let $p_3(x)$, centered at $a=0$, of $e^{-x} = f(x)$

Taylor Remainder Thm guarantees that $p_3(1) \approx e^{-1}$ with error $\le$?

$f(x) = e^{-x}$

$n = 3$

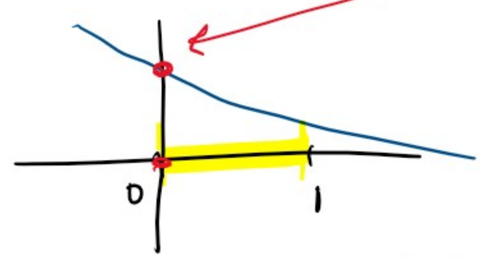
$a = 0$

$x_0 = 1$

$$|f(x_0) - p_n(x_0)| \le \frac{M\,|x_0 - a|^{n+1}}{(n+1)!}$$

$$|e^{-1} - p_3(1)| \le \frac{M\,|1-0|^4}{4!} = \frac{M}{4!} = \frac{1}{4!} = \boxed{\frac{1}{24}}$$

$$M = \max_{0 \le x \le 1} |f^{(4)}(x)| = \max_{0 \le x \le 1} |e^{-x}| = e^{-0} = 1$$



HOTSEAT: Let $p_2(x)$ be the 2nd order Taylor poly, centered at $a=1$, of $f(x) = \ln(x)$.

The estimate $p_2(2) \approx \ln(2)$ has error

$|R_2(2)| \le$?

The estimate

$$|R_2(2)| \le ?$$

(a) $\boxed{\frac{1}{3}}$

(b) $\frac{16}{3}$

(c) $\frac{8}{3}$

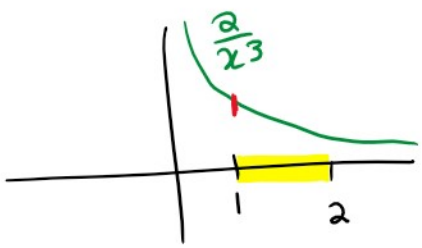
(d) $\frac{1}{12}$

$$|R_n(x_0)| \le \frac{M\,|x_0-a|^{n+1}}{(n+1)!}$$

$$\le \frac{M\,|2-1|^3}{3!} = \frac{M}{6}$$

$$M = \max_{1\le x\le 2} |f'''(x)| = \max_{1\le x\le 2}\left|\frac{-2}{x^3}\right|$$

$$= \frac{2}{1} = 2$$



$$|R_2(2)| \le \frac{2}{6} = \boxed{\frac{1}{3}}$$

S16 #9: Let $\sin(x) = \sum_{n=0}^{\infty} a_n \left(x-\frac{\pi}{6}\right)^n$ be the Taylor series for $\sin(x)$ centered at $a=\frac{\pi}{6}$

Then $a_2 = ?$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(\frac{\pi}{6}\right)}{n!}\left(x-\frac{\pi}{6}\right)^n = \sum_{n=0}^{\infty} a_n\left(x-\frac{\pi}{6}\right)^n$$

$$a_n = \frac{f^{(n)}\left(\frac{\pi}{6}\right)}{n!} \qquad n=2$$

$$a_2 = \frac{f''\left(\frac{\pi}{6}\right)}{2!} = \frac{-\sin\left(\frac{\pi}{6}\right)}{2} = \frac{-\left(\frac{1}{2}\right)}{2} = \boxed{\frac{-1}{4}}$$

$2!$ 　 2

NOTE: "What is the coefficient of $(x-\frac{\pi}{6})^2$ term?"

$a_2 = ?$

S19 #11: Given that $\arctan(x^2) = \int_0^x \frac{2t}{1+t^4}\,dt$

Find the MacLaurin series of $\arctan(x^2) = ?$

$$\arctan(x^2) = \int_0^x \frac{2t}{1+t^4}\,dt = \int_0^x 2t\left(\frac{1}{1-(-t^4)}\right)dt$$

$$= \int_0^x 2t \sum_{k=0}^{\infty} (-t^4)^k\,dt$$

$$= \int_0^x 2t \sum_{k=0}^{\infty} (-1)^k t^{4k}\,dt$$

$$= \int_0^x \sum_{k=0}^{\infty} 2(-1)^k t^{4k+1}\,dt$$

$$= \sum_{k=0}^{\infty} 2(-1)^k \int_0^x t^{4k+1}\,dt$$

$$= \sum_{k=0}^{\infty} 2(-1)^k \left[\frac{t^{4k+2}}{4k+2}\right]_0^x$$

$$= \sum_{k=0}^{\infty} \frac{\cancel{2}(-1)^k x^{4k+2}}{\underset{2}{\cancel{4}}k+\cancel{2}} = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{2k+1}}$$