Announcements:

Final Exam Mon May 2nd @ 10:30am – 12:30pm in ELLT

Office Hour @ 4-5pm in MATH 817

Warm Up: Make the appropriate u-substitution to write the integral in terms of u:

$$\int \sin^{2/3}(x)\cos^3(x)\,dx$$

(F19 #4)

(a) $\int u^{8/3}\,du$

(c) $\int \left(u^{2/3} + u^{8/3}\right)du$

$\boxed{\text{(b) } \int \left(u^{2/3} - u^{8/3}\right)du}$

$\cos^3(x) \rightarrow du = \cos(x)dx$

$u = \sin(x)$

$$\int \underset{\downarrow}{\sin^{2/3}(x)} \; \underbrace{\cos^2(x)}_{1-\sin^2(x)} \; \underset{\downarrow}{\cos(x)dx}$$

$u^{2/3}$ $\qquad$ $du$

$$\int_0^{\frac{\pi}{2}} \sin^{2/3}(x)\cos^3(x)\,dx = \int_0^1 u^{2/3}\left[1-u^2\right]du$$

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Series Questions:

F18 #15: The quantity $(\cos 2x)\sum_{n=1}^{\infty}(\tan x)^{2n}$ for $0 \le x < \frac{\pi}{4}$ is equal to ?

$$(\cos 2x)\sum_{n=1}^{\infty}(\tan^2 x)^n$$

← almost Geometric series

need n to start at 0

$k = n-1 \qquad n = k+1$

$$(\cos 2x)\sum^{\infty}(\tan^2 x)^{k+1} = (\cos 2x)(\tan^2 x)\sum(\tan^2 x)^k$$

$$(\cos 2x)\sum_{k=0}^{\infty}(\tan^2 x)^{k+1} = (\cos 2x)(\tan^2 x)\underbrace{\sum_{k=0}^{\infty}(\tan^2 x)^{k}}_{\text{Geometric}}$$

$$= \frac{(\cos 2x)(\tan^2 x)}{1-\tan^2 x}\left(\frac{\cos^2 x}{\cos^2 x}\right) = \frac{(\cos 2x)\sin^2 x}{\cos^2 x - \sin^2 x}$$

$$= \frac{(\cos 2x)\sin^2 x}{\left(\frac{\cancel{1}+\cos 2x}{2}\right) - \left(\frac{\cancel{1}-\cos 2x}{2}\right)} = \frac{\cancel{\cos 2x}\,\sin^2 x}{\frac{\cancel{2}\,\cancel{\cos 2x}}{\cancel{2}}}$$

$$= \boxed{\sin^2 x}$$

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1. Geometric series $\sum_{k=0}^{\infty} a r^k = \frac{a}{1-r}$

2. Look at other given series and put in that form

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S19 #21: Use formulas + the Alternating series estimation Theorem to compute $\int_0^{0.1} \ln(1+x)\,dx$ with an error $< 10^{-4}$

TOO LONG $\rightarrow$ Break into 2 Qs

Q1: Compute $\int_0^{0.1} \ln(1+x)\,dx$

$\int^{0.1} \sum^{\infty} (-1)^{n-1} x^{n} \, dx$

$$= \int_0^{0.1} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n!} \, dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \int_0^{0.1} x^n \, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \left[ \frac{x^{n+1}}{n+1} \right]_0^{0.1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\underbrace{n!\,(n+1)}_{(n+1)!}} \left[ \underbrace{(0.1)}_{\left(\frac{1}{10}\right)}^{n+1} - \cancel{0^{n+1}}^{\,0} \right] = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+1)!\, 10^{n+1}}}$$

HOTSEAT: Use the Alt. Series Estimation Theorem to determine how many terms you need to compute $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+1)!\, 10^{n+1}}$ with error $< 10^{-4}$

(a) 1

(b) 2

(c) 3

(d) 4

$$|R_k| \le a_{k+1} = \frac{1}{(k+2)!\, 10^{k+2}} < \frac{1}{10^4}$$

Check $k=2$ $\quad |R_2| \le \frac{1}{4!\, 10^4} < \frac{1}{10^4}$ ✓

Check $\cancel{k=1}$ $\quad |R_1| \le \frac{1}{3!\, 10^3} = \frac{1}{6 \cdot 10^3} > \frac{1}{10^4}$

$\boxed{\text{need } n=2 \text{ terms}}$

Now, we can compute

$$\int_0^{0.1} \ln(1+x)\, dx \cong \sum_{n=1}^{2} \frac{(-1)^{n-1}}{(n+1)!\, 10^{n+1}}$$

$(-1)^0 \quad (-1)^1 \quad \frac{1}{\phantom{x}} - \frac{1}{\phantom{x}}$

$$= \frac{(-1)^0}{2!\,10^2} + \frac{(-1)^1}{3!\,10^3} = \boxed{\frac{1}{200} - \frac{1}{6000}}$$

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**S19 #17** Compute the limit

$$\lim_{n\to\infty}\left(\sqrt{\underbrace{n^4+n^3+n^2}_{A}} - \sqrt{\underbrace{n^4+n^3+2n^2+1}_{B}}\right)$$

$$\lim_{n\to\infty}\left(\sqrt{A}-\sqrt{B}\right)\left(\frac{\sqrt{A}+\sqrt{B}}{\sqrt{A}+\sqrt{B}}\right) = \lim_{n\to\infty}\frac{A-B}{\sqrt{A}+\sqrt{B}}$$

$$= \lim_{n\to\infty}\frac{(\cancel{n^4}+\cancel{n^3}+n^2)-(\cancel{n^4}+\cancel{n^3}+2n^2+1)}{\sqrt{A}+\sqrt{B}} =$$

$$= \lim_{n\to\infty}\frac{-n^2-1}{\sqrt{n^4+n^3+n^2}+\sqrt{n^4+n^3+2n^2+1}} \quad \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty}\frac{-1-\frac{1}{n^2}}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+\sqrt{1+\frac{1}{n}+\frac{2}{n^2}+\frac{1}{n^4}}} = \frac{-1}{\sqrt{1}+\sqrt{1}} = \boxed{\frac{-1}{2}}$$

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## Polar Coordinates:

1. Polar $\longleftrightarrow$ Cartesian

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$

$r = f(\theta)$

2. plotting polar functions $r = f(\theta)$

plot $r$ vs. $\theta$ $\rightarrow$ plot points in the $x$-$y$ plane

3. Slope of the line tangent to $r = f(\theta)$

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = f(\theta)\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

4. Area $r = f(\theta)$ over $[\alpha, \beta]$

$$A = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\, d\theta$$

5. Arc Length $r = f(\theta)$ over $[\alpha, \beta]$

$$L = \int_\alpha^\beta \sqrt{\{f(\theta)\}^2 + \{f'(\theta)\}^2}\, d\theta$$

HOTSEAT: (S20 #11)

Find the slope of the line tangent to $r = 8\sin\theta$ at the point $(4, \frac{\pi}{6})$

(a) $\sqrt{3}$

(b) $2\sqrt{3}$

$$\frac{dy}{dx} = \left.\frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}\right|_{\theta = \frac{\pi}{6}}$$

$$f(\tfrac{\pi}{6}) = 4$$

(b) $2\sqrt{3}$

(c) $\frac{\sqrt{3}}{2}$

$f\left(\frac{\pi}{6}\right) = 4$

$f'\left(\frac{\pi}{6}\right) = 8\cos\left(\frac{\pi}{6}\right) = 8 \cdot \frac{\sqrt{3}}{2} = 4\sqrt{3}$

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$$\frac{dy}{dx} = \frac{(4\sqrt{3})\left(\frac{1}{2}\right) + (4)\left(\frac{\sqrt{3}}{2}\right)}{(4\sqrt{3})\left(\frac{\sqrt{3}}{2}\right) - (4)\left(\frac{1}{2}\right)} = \frac{4\sqrt{3}}{6-2} = \boxed{\sqrt{3}}$$