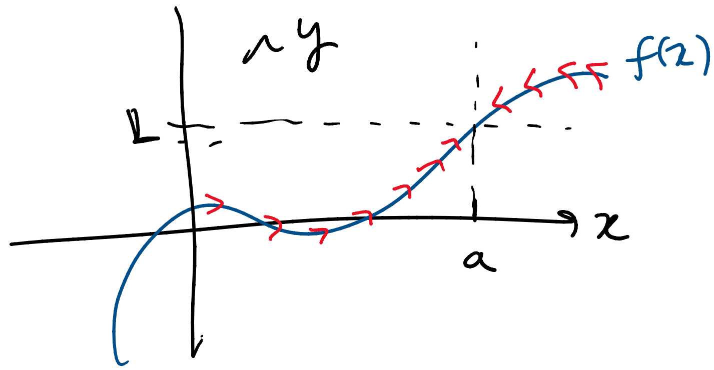
Lesson 10 - 9/13/2023 - 4:30pm

15.2: Limits & Continuity

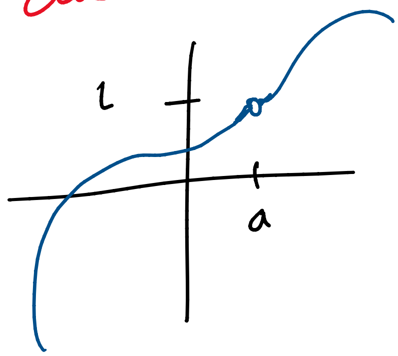
Recall in Calc

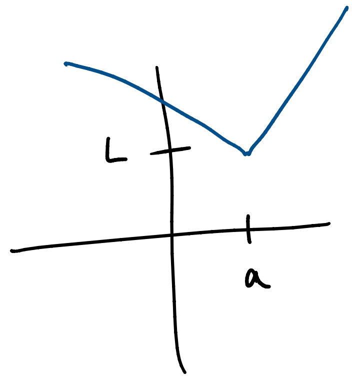
$$\lim_{x \to a} f(x) = L$$

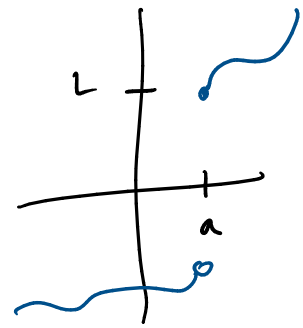


as x gets closer to a
$f(x)$ gets closer to L

Examples:







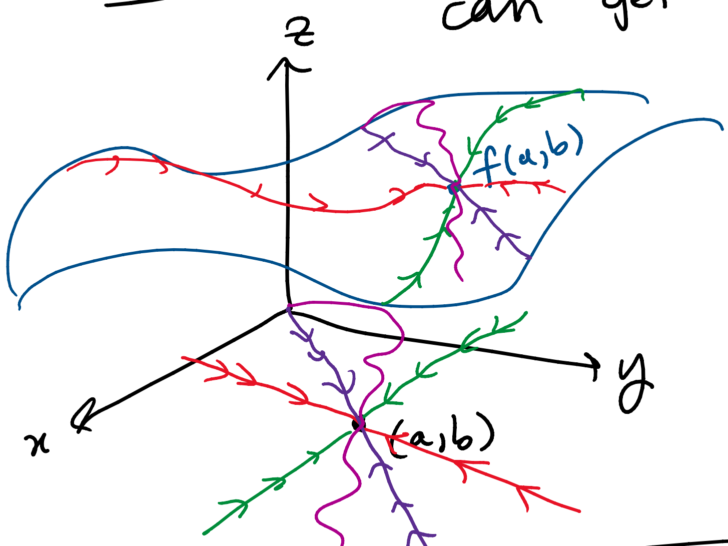
DNE
Does Not Exist

In multiple variables:

$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$

if as (x,y) gets closer to (a,b)
then $f(x,y)$ gets closer to L

NOTE: There are many ways that (x,y) can get close to (a,b)



path $x = a$

path $y = b$

path $y = b + (x - a)$

path *

For limit to exist it should not depend

For a limit to exist, it should not depend on the path you take

Example:

$$\lim_{(x,y)\to(0,0)} \sin(xy)$$

$\sin(\theta)$ where $\theta = xy$

$\sin(\theta)$ is continuous everywhere

$u = xy$

$$= \lim_{u\to 0} \sin(u) = \sin(0) = 0$$

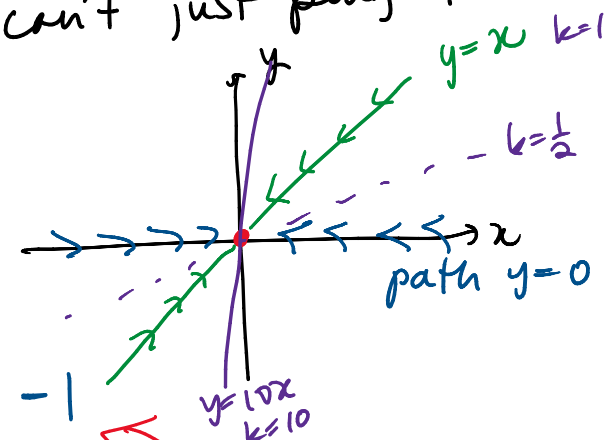
If $\sin(\theta)$ is continuous $\lim_{\theta\to\theta_0} \sin(\theta) = \sin(\theta_0)$

Example:

$$\lim_{(x,y)\to(0,0)} \frac{y^2-x^2}{x^2+y^2}$$

Not defined at $(0,0)$ can't just plug in

Check different paths



$\boxed{\text{path 1 } y=0}$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} \frac{y^2-x^2}{x^2+y^2} = \lim_{x\to 0} \frac{0^2-x^2}{x^2+0^2} = -1$$

Two values along 2 different paths

$\boxed{\text{DNE}}$

$\boxed{\text{path 2 } y=x}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{y^2-x^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-x^2}{x^2+x^2} = 0$$

path $y=kx$ where k is number

$$\lim_{(x,y)\to(0,0)} \frac{y^2-x^2}{x^2+y^2} = \lim_{x\to 0} \frac{(kx)^2-x^2}{x^2+(kx)^2} = \lim_{x\to 0} \frac{(k^2-1)\cancel{x^2}}{(1+k^2)\cancel{x^2}}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=kx}} \frac{y^2-x^2}{x^2+y^2} = \lim_{x\to 0} \frac{(kx)^2-x^2}{x^2+(kx)^2} = \lim_{x\to 0} \frac{(k^2-1)\cancel{x^2}}{(1+k^2)\cancel{x^2}}$$

$$= \frac{k^2-1}{1+k^2}$$

dependence on k

$\Longrightarrow$ dependence on path

$\boxed{\text{DNE}}$

k	path	L
1	$y=x$	$\frac{1^2-1}{1+1^2}=0$
2	$y=2x$	$\frac{2^2-1}{1+2^2}=\frac{3}{5}$
10	$y=10x$	...

Example:

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx}} \frac{y^2-x^2}{x+y} = \lim_{x\to 0} \frac{(kx)^2-x^2}{x+kx} = \lim_{x\to 0} \frac{(k^2-1)x^{\cancel{2}}}{(1+k)\cancel{x}}$$

$$= \frac{(k^2-1)}{(1+k)} \lim_{x\to 0} x = \left(\frac{k^2-1}{1+k}\right)\cdot 0$$

$= 0$ for all k

Better Way:

$$\lim_{(x,y)\to(0,0)} \frac{(y-x)\cancel{(y+x)}}{\cancel{x+y}} = \lim_{(x,y)\to(0,0)} y-x = 0-0 = 0$$

☆ Continuity:

Def: The function f is continuous at the point (a,b) if

1. f is defined at (a,b)
2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists

2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists

3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$