Lesson 11 - 9/15/2023 - 3:30pm

15.3: Partial Derivatives

WARM UP: $\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2} = \lim_{x\to 0} \frac{(x+kx)^2}{x^2+(kx)^2}$

along $y=kx$

$= \lim_{x\to 0} \frac{(1+k)^2\cancel{x^2}}{(1+k^2)\cancel{x^2}} = \frac{(1+k)^2}{1+k^2} = \frac{1+2k+k^2}{1+k^2}$

k - slope

k	line	L
0	$y=0$	1
1	$y=x$	$\frac{(1+1)^2}{1+1^2} = \frac{4}{2} = 2$
π	$y=\pi x$	$\frac{(1+\pi)^2}{1+\pi^2}$

If L depends on k, then L is different for each path

$\boxed{\text{DNE}}$

☆ Derivatives:

one variable $g(x)$

$$g'(x) = \frac{dg}{dx} = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

- instantaneous rate of change

two variables: $f(x,y)$

- f can change in x and in y

• two derivatives

partial derivative with respect to x

curly ∂ partial

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Treat y as a constant

partial derivative with respect to y

$$\frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Example: $f(x,y) = 4x^2 + 2xy - y^2$

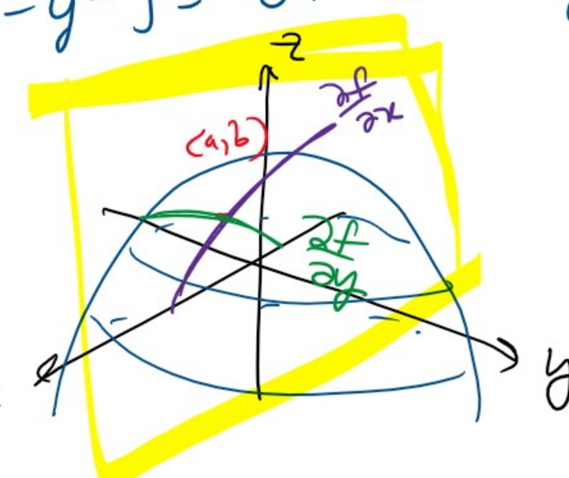
Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

keep y constant

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[4x^2 + 2xy - y^2\right]$$

$$= 8x + 2y + 0$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[4x^2 + 2xy - y^2\right] = 0 + 2x - 2y$$



2nd order Partial Derivatives

$$\begin{tikzcd}[column sep=small]
& & & f(x,y) \arrow[dll] \arrow[drr] & & & \\
& \frac{\partial f}{\partial x} \arrow[dl] \arrow[dr] & & & & \frac{\partial f}{\partial y} \arrow[dl] \arrow[dr] & \\
\frac{\partial^2 f}{\partial x^2} & & \frac{\partial^2 f}{\partial y \partial x} & & \frac{\partial^2 f}{\partial x \partial y} & & \frac{\partial^2 f}{\partial y^2}
\end{tikzcd}$$

Example: $f(x,y) = 3e^{xy}$

1st $f_x = \frac{\partial f}{\partial x} = 3ye^{xy}$ $\qquad f_y = \frac{\partial f}{\partial y} = 3xe^{xy}$

2nd: $f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[3ye^{xy}\right] = 3y^2 e^{xy}$

$$f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left[3ye^{xy}\right] = 3e^{xy} + 3yxe^{xy}$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left[3xe^{xy}\right] = 3e^{xy} + 3xye^{xy}$$

Def: f_{xy} and f_{yx} are called <u>mixed partial derivatives</u>

Thm: (Clairaut) Suppose $f(x,y)$ is defined at (a,b) in its Domain D. If f_{yx} and f_{xy} are continuous on D then

$$f_{xy} = f_{yx}$$

Partial Differential Equations: an equation that relates the partial derivatives of a function

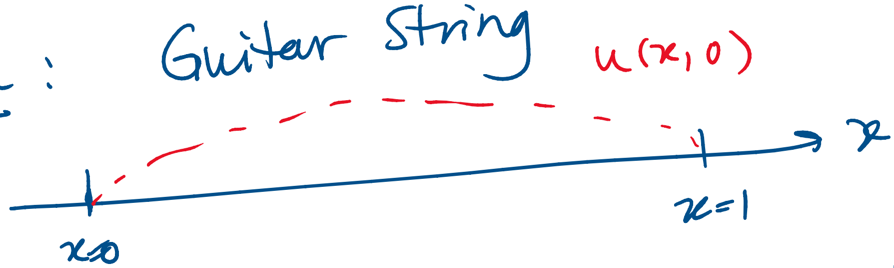
Examples:

$$u_t = c^2 (u_{xx} + u_{yy}) \quad \text{heat eqn.}$$

$$u_{tt} = c^2 (u_{xx} + u_{yy}) \quad \text{wave eqn.}$$

$$u_{xx} + u_{yy} = 0 \quad \text{Laplace's eqn}$$

Example: Guitar String



Let $u(x,t)$ be displacement of string

$$u_{tt} = c^2 u_{xx} \quad \text{1 dim wave eqn.}$$

Check soln: $u(x,t) = 3\cos(2\pi t)\sin(\pi x)$

$$u_t = -3 \cdot 2\pi \sin(2\pi t)\sin(\pi x)$$

$$u_{tt} = -3 \cdot (2\pi)^2 \cos(2\pi t)\sin(\pi x)$$

$$u_x = 3\pi \cos(2\pi t)\cos(\pi x)$$

$$u_{xx} = -3\pi^2 \cos(2\pi t)\sin(\pi x)$$

$$u_{tt} = 4 u_{xx} = 2^2 u_{xx} \quad \checkmark$$