Lesson 12 - 9/18/2023 - 4:30pm

15.4: The Chain Rule

Intuition: Mixed partials

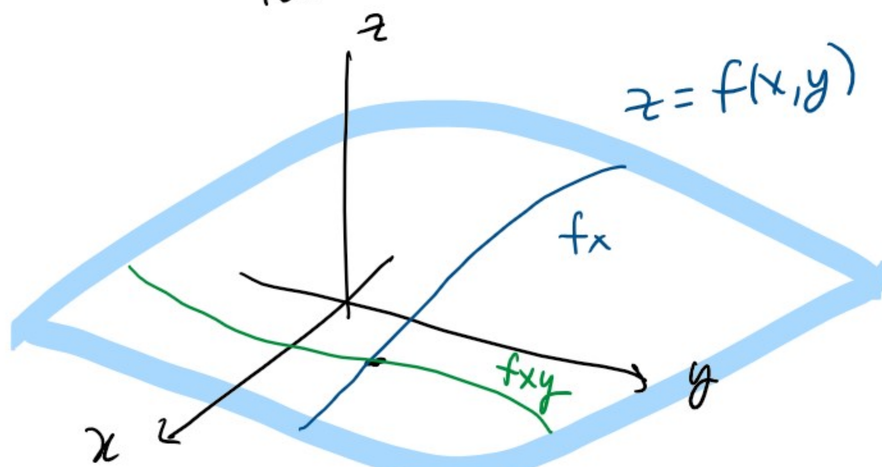
$z = f(x,y)$

$f_x = \lim_{h \to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$ – step in x → look at slop

f_y – – '' '' y → '' ''

f_{xx} – rate of change of slope in x

f_{xy} – take a step in x
look at slope in y



NOTATION:

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = f_{xy}$$

The variable closest to f is taken deriv first

Take x-deriv first

Take y deriv 2nd

Chain Rule:

$f(x)$ and $x(t) \to f(x(t))$

Chain Rule:

Calc $f(x)$ and $x(t) \to f(x(t))$

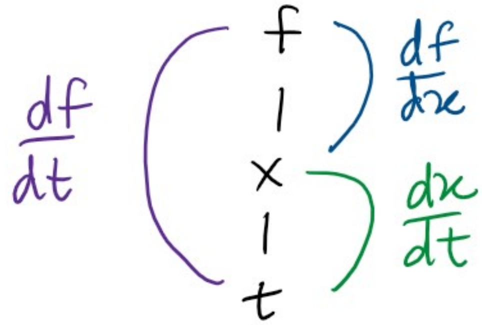
$\underline{Q}$: $\dfrac{df}{dt} = \dfrac{df}{dx} \cdot \dfrac{dx}{dt}$

pretend the derivatives are fractions
dx terms cancel out

Here: x – indep var $f(x)$
dep var $x(t)$
– dissappears after chain rule
– call it intermediate variable

It's useful to draw a tree diagram

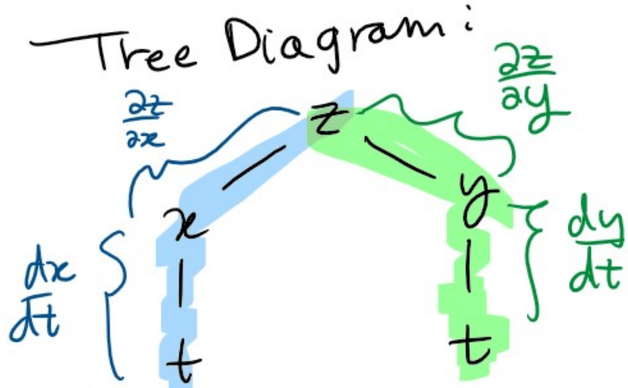
each step down is a derivative



Calc 3: $z = f(x, y)$
and $x = x(t)$ and $y = y(t)$
$z(t) = f(x(t), y(t))$

Q: What is $\dfrac{dz}{dt}$?

Tree Diagram:



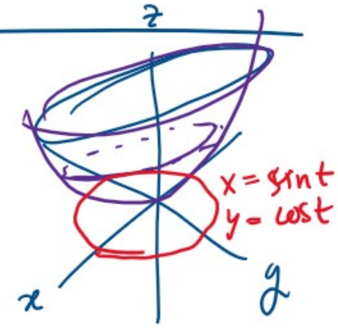
$$\boxed{\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial z}{\partial y} \cdot \dfrac{dy}{dt}}$$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}$$

Example:

$z = f(x,y) = 4x^2 + 3y^2$

Elliptic Parabola



$x = \sin(t)$ $\quad y = \cos(t)$

circle of radius one

Q: $\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$

$= (8x)\frac{dx}{dt} + (6y)\frac{dy}{dt}$

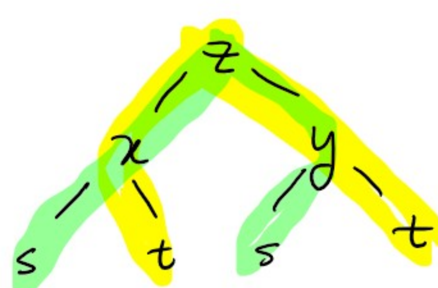
$= 8\sin(t)\cos(t) + 6\cos t(-\sin t)$

$= 8\sin(t)\cos(t) - 6\sin(t)\cos(t) = \boxed{2\sin(t)\cos(t) = \frac{dz}{dt}}$

Q: What if the intermediate values x and y were functions of two variables?

$x = x(s,t)$ and $y = y(s,t)$

$z = f(x,y)$



How do we find $\frac{\partial z}{\partial s}$?

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Implicit Differentiation:

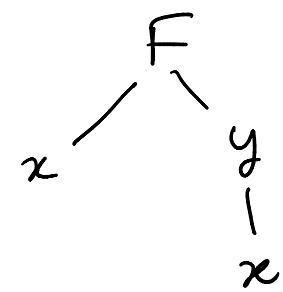
Suppose $F(x,y) = 0$ defines $y = y(x)$ implicitly

Ex: $F(x,y) = \cos(xy) + 3y^2 - x = 0$

x - indep. var

y - dep var.

Q: What is $\frac{dy}{dx}$?



$$\frac{d}{dx}\left[F(x,y) = 0\right]$$

$$\frac{\partial F}{\partial x}\cdot\cancelto{1}{\frac{dx}{dx}} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$$

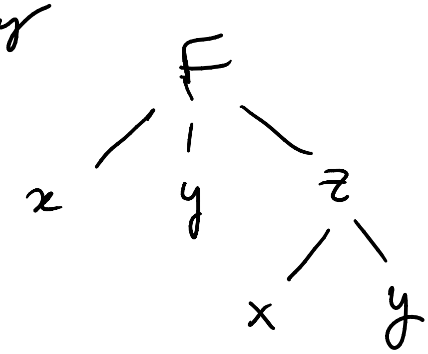
$$\frac{d}{dx}[x] = 1$$

$$\boxed{\frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-F_x}{F_y}}$$

Calc 3: Suppose $F(x,y,z) = 0$ defines $z = z(x,y)$ implicitly

x, y - indep. var

z - dep var.



Then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$?

Then $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \quad y$

Q: $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$?

$$\frac{\partial}{\partial x}\left[F(x,y,z) = 0\right]$$

$$\frac{\partial F}{\partial x}\cdot \underbrace{\cancel{\frac{\partial x}{\partial x}}}_{1} + \frac{\partial F}{\partial y}\cdot \underbrace{\cancel{\frac{\partial y}{\partial x}}}_{0} + \frac{\partial F}{\partial z}\cdot \frac{\partial z}{\partial x} = 0$$

$\frac{\partial}{\partial x}[x] = 1 \qquad \frac{\partial}{\partial x}[y] = 0$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = \frac{-F_x}{F_z}}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}}$$