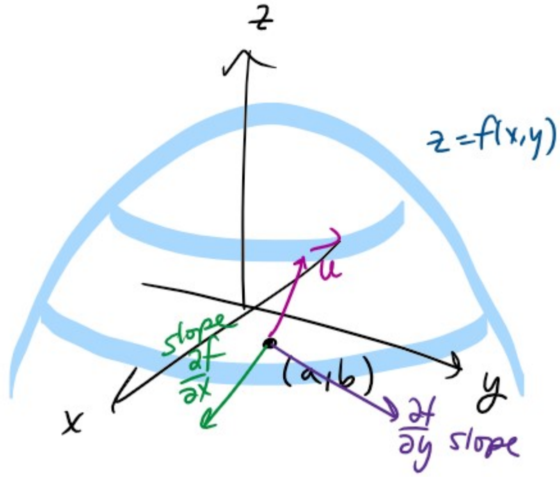
Lesson 13 - 9/20/2023 - 4:30pm

**15.5: Directional Derivatives & the Gradient**




Directional Derivatives:

Given $z = f(x,y)$

The partial derivatives $f_x$ and $f_y$ represent the slope in $\hat{i}$ and $\hat{j}$ respectively

Q: What is the slope in direction $\vec{u}$?

$\vec{u} = \langle u_1, u_2 \rangle$

want a unit vector: $|\vec{u}| = 1 = \sqrt{u_1^2 + u_2^2}$

Def: The directional derivative at $(x,y) = (a,b)$ in the direction $\vec{u} = \langle u_1, u_2 \rangle$ (where $|\vec{u}| = 1$) is:

$$D_{\vec{u}} f(a,b) = \lim_{h \to 0} \frac{f(a + h u_1, b + h u_2) - f(a,b)}{h}$$

$\vec{u} = \langle 1, 0 \rangle = \hat{i} \longrightarrow \frac{\partial f}{\partial x}$

$\vec{u} = \langle 0, 1 \rangle = \hat{j} \longrightarrow \frac{\partial f}{\partial y}$

Let's define a function

$x(t) = a + t u_1$

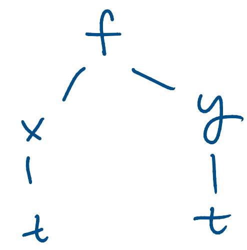
$y(t) = b + t u_2$

$g(t) = f(x(t), y(t))$

$$g'(0) = \lim_{t \to 0} \frac{g(t) - g(0)}{t} = \lim_{t \to 0} \frac{f(x(t), y(t)) - f(x(0), y(0))}{t}$$

$$= \lim \frac{f(a + t u_1, b + t u_2) - f(a,b)}{t} = D_{\vec{u}} f(a,b)$$

$$= \lim_{t \to 0} \frac{f(a+tu_1, b+tu_2) - f(a,b)}{t} = D_{\vec{u}} f(a,b)$$



$$= \frac{d}{dt}\left[ f(x(t), y(t)) \right]\Big|_{t=0}$$

$$= \frac{\partial f}{\partial x}^{(x(t),y(t))} \frac{dx}{dt} + \frac{\partial f}{\partial y}^{(x(t),y(t))} \cdot \frac{dy}{dt} \Big|_{t=0}$$

$$\frac{dx}{dt} = \frac{d}{dt}\left[ a + tu_1 \right] = u_1$$

$$= \frac{\partial f}{\partial x} \cdot u_1 + \frac{\partial f}{\partial y} \cdot u_2 \Big|_{t=0}$$

$$= \frac{\partial f}{\partial x}(a,b) \cdot u_1 + \frac{\partial f}{\partial y}(a,b) \cdot u_2$$

$$\boxed{D_{\vec{u}} f(a,b) = \frac{\partial f}{\partial x}(a,b) \cdot u_1 + \frac{\partial f}{\partial y}(a,b) \cdot u_2}$$

$\vec{u} = \langle u_1, u_2 \rangle$

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## Gradient:

**Def:** The gradient of $f(x,y)$ is

$$\vec{\nabla} f(x,y) = \langle f_x, f_y \rangle$$
$$= f_x \hat{\imath} + f_y \hat{\jmath}$$

$\vec{\nabla}$ – "del" "nabla"

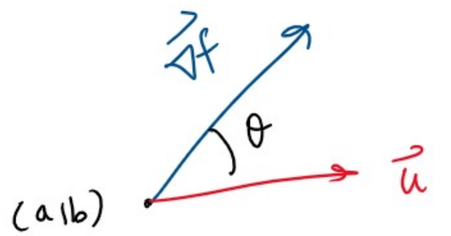
**Notation:** $\vec{\nabla} f = \nabla f = \text{grad}\, f$

**NOTE:** $\vec{\nabla} f$ is a vector

## Directional Derivatives:

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = \frac{\partial f}{\partial x} \cdot u_1 + \frac{\partial f}{\partial y} \cdot u_2$$

$$D_{\vec{u}} f = \underbrace{\vec{\nabla} f \cdot \vec{u}}_{\text{dot product}} = \frac{\partial f}{\partial x}\cdot u_1 + \frac{\partial f}{\partial y}\cdot u_2$$



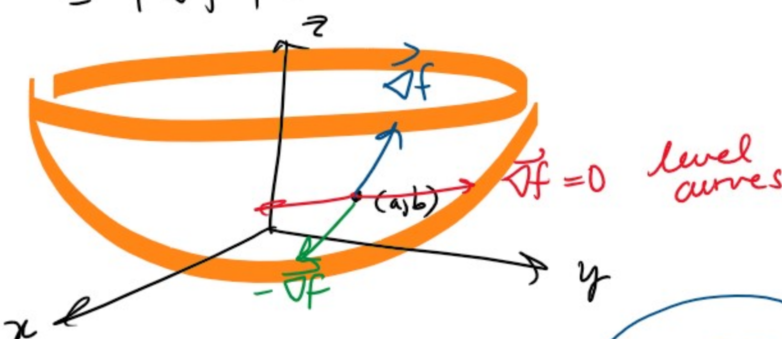

at the point (a,b)

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = |\vec{\nabla} f| \cdot \cancel{|\vec{u}|}^{\,1} \cos\theta = |\vec{\nabla} f| \cos\theta$$

| $\theta$ | $D_{\vec{u}} f$ | Note | Name |
|---|---|---|---|
| $0$ | $\lvert\vec{\nabla} f\rvert$ | $\vec{u}$ is parallel to $\vec{\nabla} f$ | Direction of steepest ascent |
| $\frac{\pi}{2}$ | $0$ | $\vec{u}$ is orthogonal to $\vec{\nabla} f$ | zero change in $f$; $f(x,y) = c$; level curves |
| $\pi$ | $-\lvert\vec{\nabla} f\rvert$ | $\vec{u}$ is antiparallel to $\vec{\nabla} f$ | Direction of Steepest Descent |

NOTE: $\vec{\nabla} f$ is orthogonal to level curves

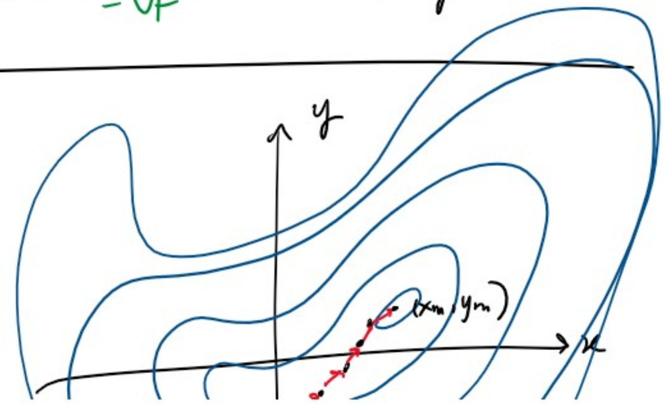
$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = |\vec{\nabla} f| \cos\theta = scal_{\vec{u}}(\vec{\nabla} f)$$



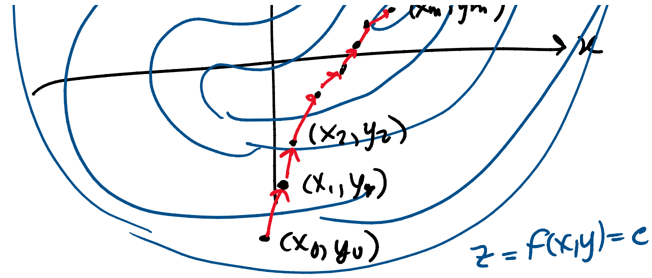

## Gradient Descent:

$z = f(x,y)$

Want to find the



Want to find the minimum of $z$ at $(x_m, y_m)$




Starting point $(x_0, y_0)$

1. Calculate the direction of steepest descent.
$$\vec{u} = -\nabla f$$
2. Take a step in that direction to get $(x_i, y_i)$
3.