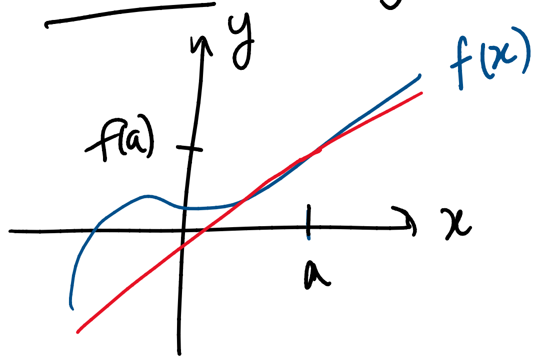
15.6: Tangent Planes and Linear Approximation

Tangent Plane:

Calc 1: Tangent Line



Tangent Line:

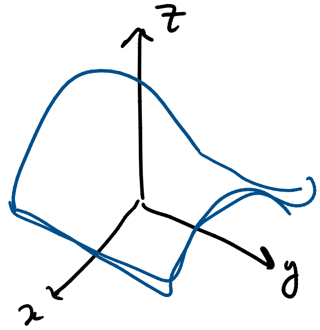
$$y = f(a) + f'(a)(x-a)$$

Calc 3: Tangent Plane to a Surface

A surface in 3D can be defined either

Explicitly:

$$z = f(x,y)$$



Implicitly

$$F(x,y,z) = 0$$

Ex: $z = -x^2 + y^2$

$$z + x^2 - y^2 = 0$$

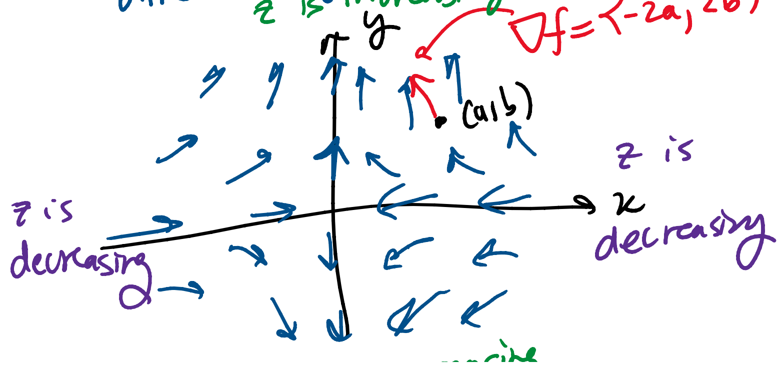
$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle -2x, 2y \rangle$$

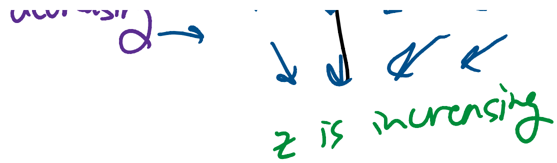
direction of greatest increase

$$\vec{\nabla} F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle = \langle 2x, -2y, 1 \rangle$$

also a vector field in 3D

$\rightarrow$ Slides.





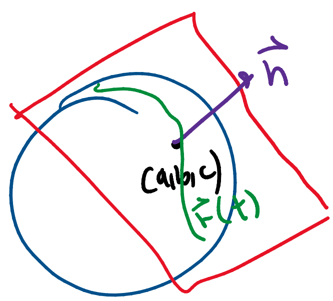
Q: At a point (a,b,c) on the surface $F(x,y,z)=0$

What is the tangent plane?

plane — Point (a,b,c)

— Normal vector $\vec{n}=\langle n_1, n_2, n_3\rangle$

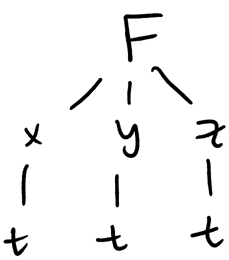
Equation of the plane : $\vec{n}\cdot\langle x-a,\, y-b,\, z-c\rangle = 0$



Assuming $\vec{r}(t)=\langle x(t), y(t), z(t)\rangle$ lies on the surface

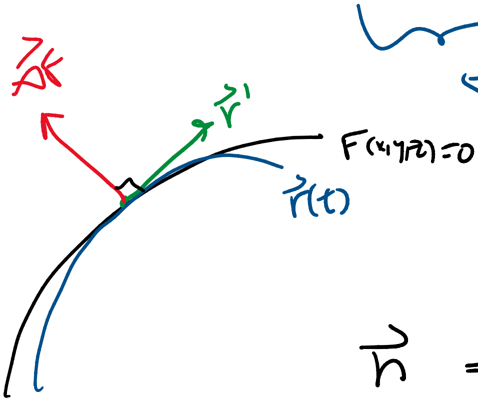
$\vec{r}(t)$ satisfies $F(x,y,z)=0$

$$\frac{d}{dt}\Big[F(x(t), y(t), z(t)) = 0\Big] \leftarrow \text{curve } \vec{r}(t) \text{ lies on surface}$$



$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial F}{\partial z}\cdot\frac{dz}{dt} = 0$$

$$\underbrace{\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right\rangle}_{\vec{\nabla} F} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle}_{\substack{\vec{r}\,'(t) \\ \text{tangent vector}}} = 0$$



$$\vec{\nabla} F \cdot \vec{r}\,'(t) = 0$$

$\vec{n} = \vec{\nabla} F$ is orthogonal to tangent vector $\vec{r}\,'(t)$

Equation of Tangent Plane: … $= 0$

Equation of Tangent Plane:

$$\vec{n} \cdot \langle x-a,\ y-b,\ z-c \rangle = 0$$

$$\vec{\nabla} F \cdot \langle x-a,\ y-b,\ z-c \rangle = 0$$

$$F_x(x-a) + F_y(y-b) + F_z(z-c) = 0$$

Linear Approximation:

Calc 1:

when x is close to a

$$f(x) \approx f(a) + f'(a)(x-a)$$

Calc 3: $z = f(x,y)$ [Explicit Definition]

when (x,y) is close to (a,b), we can approximate

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Example: $f(x,y) = e^x \cos(y)$

Find a linear approximation near $(0,0)$

$$L(x,y) \simeq f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$

$$f(0,0) = e^0 \cos(0) = 1$$

$$f_x(0,0) = e^x \cos(y) \Big|_{x=0,\ y=0} = 1$$

$f(0,0) = 1$

$$f_x(0,0) = e^x\cos(y)\Big|_{\substack{x=0\\y=0}} = 1$$

$$f_y(0,0) = -e^x\sin(y)\Big|_{\substack{x=0\\y=0}} = 0$$

$$L = 1 + x + 0\cdot y = \boxed{1+x}$$

Differentials:

Def: Let (a,b) be in the domain D of $f(x,y)$. Choose Δx and Δy so that $(a+\Delta x, b+\Delta y)$ is in D.

Then, the total differential df of $f(x,y)$ at (a,b) is:

$$df = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$

Example: Find df of $f(x,y) = 3x^2 - 2xy + y^2$ and use it to approximate Δf at $(2,-3)$ use $\Delta x = 0.1$ and $\Delta y = -0.05$

$$df = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$

$$f_x = 6x - 2y\Big|_{\substack{x=2\\y=-3}} = 6\cdot 2 - 2\cdot(-3) = 18$$

$$f_y = -2x + 2y\Big|_{\substack{x=2\\y=-3}} = -2\cdot 2 + 2(-3) = -10$$

$$df = 18\cdot 0.1 + (-10)(-0.05)$$

$$= 1.8 + 0.5 = \boxed{2.3 = df}$$