Lesson 17 - 9/29/2023 - 4:30pm

15.8: Lagrange Multipliers

WARM UP:

c.p. $(0,0)$, $(1,-1)$, $(-1,-1)$

$f_x = 2x(y^3-y)$

$f_y = x^2(3y^2-1)+2y$

2nd Deriv Test

$f_{xx} = 2(y^3-y)$ $\quad f_{xy} = 2x(3y^2-1)$

$f_{yy} = (x^2)(6y)+2$

$\boxed{(0,0)}$ $\quad f_{xx} = 0 \quad f_{xy} = 0 \quad f_{yy} = 2$

$D = 0\cdot 2 - 0^2 = 0 \rightarrow$ undetermined

$\boxed{(1,-1)}$ $\quad f_{xx} = 2((-1)^3-(-1)) = 0 \quad f_{xy} = 2\cdot 1(3\cdot(-1)^2-1) = 2\cdot 2 = 4$

$f_{yy} = (1)^2 6(-1)+2 = -4$

$D = 0\cdot(-4) - 4^2 = -16 < 0 \rightarrow$ saddle point

$(-1,-1)$ $\quad f_{yy} = -4 \quad f_{xy} = -4 \quad f_{xx} = 0$

$D = (0)(-4) - (-4)^2 < 0 \rightarrow$ saddle point

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*Optimization:

Find the max/min of $f(x,y)$ subject to a constraint $g(x,y)=0$

that makes golf balls

a constr

Example: A company that makes golf balls

x – # of golf balls

y – # of ads purchases

$f(x,y)$ — profit function $\longrightarrow$ max f

$g(x,y)=0$ – budgetary constraint

$$\underbrace{\max \underbrace{f(x,y)}_{\text{objective}} \text{ subject to } \underbrace{g(x,y)=0}_{\text{constraint}}}_{\text{optimization problem}}$$

Ex: max $f(x,y) = -x^2 - 2xy - 9y^2 + 48x + 96y$

subject to $5x + y - 54 = 0$

$f(x,y)$ — surface in 3D

$g(x,y)=0$ — curve in 2D

Thm: Assume f restricted $g(x,y)=0$ has a local extremum at (a,b) and $\nabla g(a,b) \neq 0$. Then there is a number λ called the **Lagrange multiplier** for which

$$\nabla f(a,b) = \lambda \underbrace{\nabla g(a,b)}_{\text{vector}}$$

$$\underbrace{\nabla f}_{\text{vector } \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle}(a,b) = \lambda \underbrace{\nabla g}_{\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \rangle \text{ vector}}(a,b)$$

$$\vec{\nabla} f \parallel \vec{\nabla} g$$

Why?: Parameterize $g(x,y)=0$

$\vec{r}(s) = \langle x(s), y(s) \rangle$

s – arclength

$x(0)=a$

$y(0)=b$

Then $f(x(s), y(s))$ is a function of s

@ $s=0$ $\quad x=a, \quad y=b, \quad \frac{\partial f}{\partial s} = 0$ ← because (a,b) is a local extremum

Using chain Rule

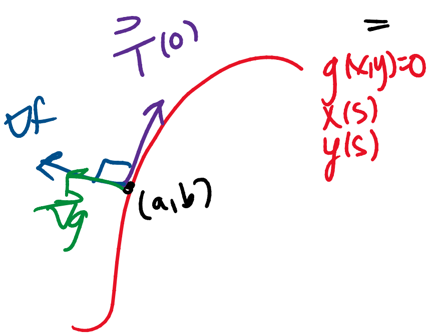
$$\begin{array}{ccc} & f & \\ x & & y \\ | & & | \\ s & & s \end{array}$$

$$0 = \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= \underbrace{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle}_{\vec{\nabla} f} \cdot \underbrace{\left\langle \frac{\partial x}{\partial s}, \frac{\partial y}{\partial s} \right\rangle}_{\vec{T}(0)}$$

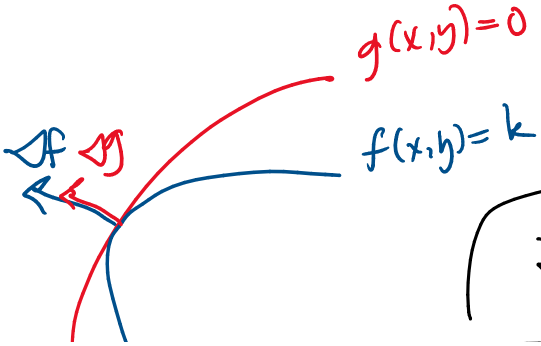
tangent vector $\vec{r}(s)$

$\vec{T}(s) = \vec{r}\,'(s)$



$$\vec{\nabla} f \perp \vec{T}(0)$$

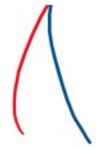
$$\vec{\nabla} g \perp \vec{T}(0)$$



$$\vec{\nabla} f \parallel \vec{\nabla} g$$ ← scalar multiples

$$\boxed{\vec{\nabla} f = \lambda \vec{\nabla} g}$$

λ – Lagrange mult.



$$\boxed{\nabla f = \lambda \nabla g} \quad \text{mult.}$$

$g(x,y)=0$ is tangent to the maximal level curve

Ex: max $-x^2 - 2xy - 9y^2 + 48x + 96y$

subject $5x + y - 54 = 0$

Lagrange Multipliers

$$\nabla f = \lambda \nabla g$$

$$\langle -2x - 2y + 48, -2x - 18y + 96 \rangle = \lambda \langle 5, 1 \rangle$$

3 eqns
3 unknowns

$$\begin{cases} -2x - 2y + 48 = 5\lambda \\ -2x - 18y + 96 = \lambda \\ 5x + y - 54 = 0 \end{cases}$$

2 equations
3 unknowns
x, y, λ

$$-2x - 2y + 48 = 5\lambda = 5(-2x - 18y + 96)$$

$$8x = 432 - 88y$$

$$x = 54 - 11y$$

$$5x + y = 54$$

$$5(54 - 11y) + y = 54$$

$$-55y + y = 54 - 5 \cdot 54$$

$$-54y = -4 \cdot 54$$

$$\boxed{y = 4}$$

$$x = 54 - 11y$$

$$\boxed{x = 54 - 44 = 10}$$

optimal @ $(10, 4) \longrightarrow f(10,4) = 540$

optimal @ (10,4)

(Sp 23 E2 #3)

Find the max of $f(x,y,z) = x+y+z$

Subj. to $(x-1)^2 + y^2 + z^2 = 1$

$$\nabla f = \lambda \nabla g$$

$$\langle 1, 1, 1 \rangle = \lambda \langle 2(x-1), 2y, 2z \rangle$$

$1 = 2\lambda(x-1)$ $\quad$ $1 = 2\lambda y$ $\quad$ $1 = 2\lambda z$

$x = 1 + \frac{1}{2\lambda}$ $\quad$ $y = \frac{1}{2\lambda}$ $\quad$ $z = \frac{1}{2\lambda}$

$$(x-1)^2 + y^2 + z^2 = 1$$

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1$$

$$3 = 4\lambda^2$$

$$\lambda = \pm\frac{\sqrt{3}}{2}$$

$x = 1 \pm \frac{1}{2\frac{\sqrt{3}}{2}} = 1 \pm \frac{1}{\sqrt{3}}$

$y = \pm\frac{1}{\sqrt{3}}$

$z = \pm\frac{1}{\sqrt{3}}$

$$f(x,y,z) = 1 \pm \frac{1}{\sqrt{3}} \pm \frac{1}{\sqrt{3}} \pm \frac{1}{\sqrt{3}} = 1 \pm \frac{3}{\sqrt{3}}$$
$$= 1 \pm \sqrt{3}$$

max at $x = 1 + \frac{1}{\sqrt{3}}$, $y = \frac{1}{\sqrt{3}} = z$ $\quad$ $f(\ldots) = 1 + \sqrt{3}$