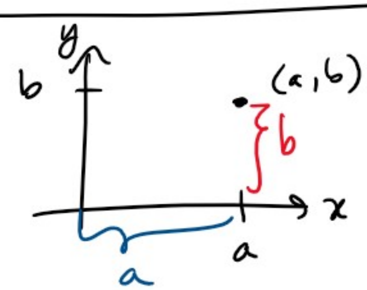
Lesson 20 - 10/13/2023 - 4:30pm

16.3: Double Integrals in Polar Coordinates

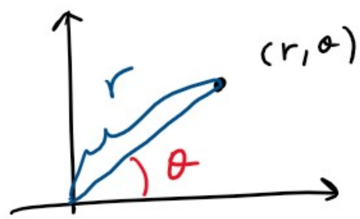
Cartesian (x, y) | Polar (r, θ)



Convert

$x = r\cos\theta$

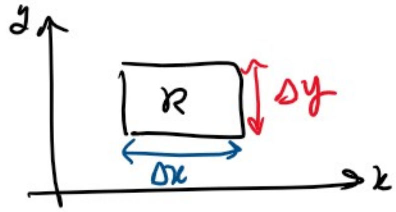
$y = r\sin\theta$



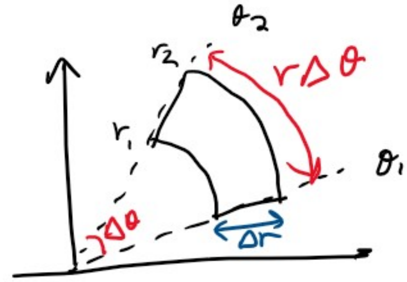
Convert

$r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$



$\Delta A = \Delta x \Delta y$



$\Delta A \approx \Delta r (r \Delta\theta)$

$\Delta A = r \Delta r \Delta\theta$

$$\iint_R f(x,y)\, dA = \iint f(x,y)\, dx\, dy$$

$$\iint_R f(r,\theta)\, dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\, \underbrace{r}_{\text{extra } r \text{ term}}\, dr\, d\theta$$

Polar "Rectangular" Region:

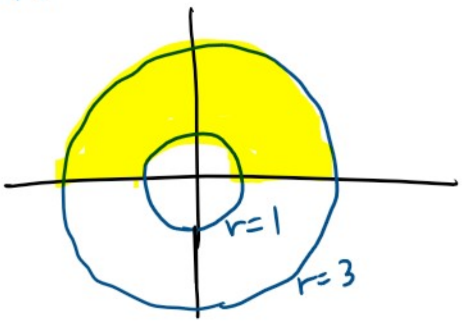
$R = \{ (r,\theta) : a \le r \le b, \; c \le \theta \le d \}$

Example: Compute

$\iint (9 - x^2 - y^2)\, dA$ where R is the region where $y \ge 0$

between curves

$$\iint_R (9-x^2-y^2)\,dA$$ where $y \geq 0$ between curves $x^2+y^2=9$ and $x^2+y^2=1$



$r^2=9$

$r=3$

$R = \{(r,\theta) : 1 \leq r \leq 3,\ 0 \leq \theta \leq \pi\}$

$$f(x,y) = 9 - x^2 - y^2$$
$$= 9 - (x^2+y^2) = 9 - r^2$$

$$\int_0^{\pi}\int_1^3 (9-r^2)\, r\,dr\,d\theta = \int_0^{\pi}\int_1^3 (9r - r^3)\,dr\,d\theta$$

$$= \int_0^{\pi} \left[\frac{9r^2}{2} - \frac{r^4}{4}\right]_1^3 d\theta = \int_0^{\pi}\left[\left(\frac{9\cdot 9}{2} - \frac{81}{4}\right) - \left(\frac{9}{2} - \frac{1}{4}\right)\right] d\theta$$

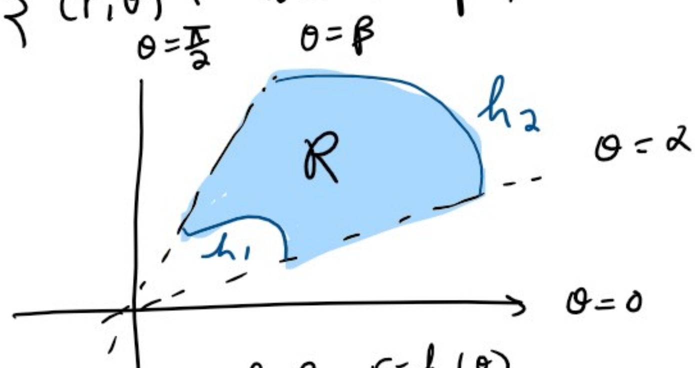
$$= \left(\frac{81}{4} - \frac{17}{4}\right)\int_0^{\pi} d\theta = 16\,[\theta]_0^{\pi} = \boxed{16\pi}$$

$16 = \frac{64}{4}$

General Polar Regions

Most common to have Type I regions

$R = \{(r,\theta) : \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$

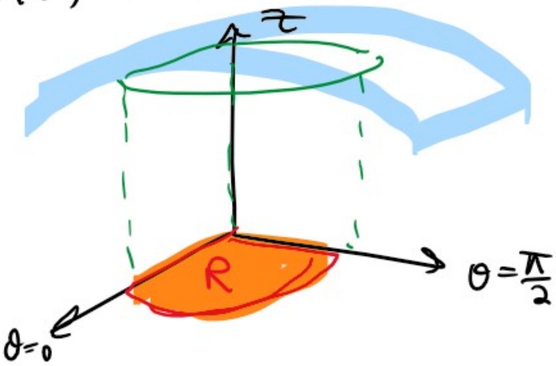


$$\iint f(r,\theta)\,dA = \int^{\theta=\beta} \int^{r=h_2(\theta)} f(r,\theta)\, r\,dr\,d\theta$$

$$\iint_R f(r,\theta)\,dA = \int_{\theta=\alpha} \int_{r=h_1(\theta)} f(r,\theta) \ldots$$

☆ Volume :

Find the volume of the solid that lies under the surface $z = f(r,\theta)$ and above R



$$V = \iint_R f(r,\theta)\, r\,dr\,d\theta$$

☆ Area : The area of region R

$$A = \iint_R 1 \cdot r\,dr\,d\theta$$

$$A = \int_0^{\frac{\pi}{4}} \int_0^{2\cos\theta} r\,dr\,d\theta = \int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta$$

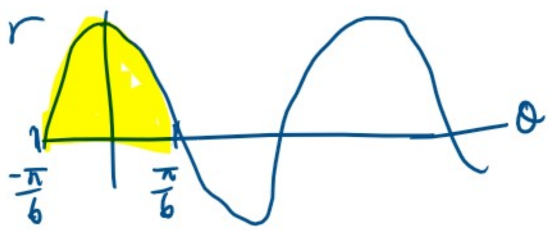
$$= \int_0^{\frac{\pi}{4}} \frac{\cancel{4}^{\,2}\cos^2\theta}{\cancel{2}}\,d\theta = \int_0^{\frac{\pi}{4}} 1 + \cos(2\theta)\,d\theta$$

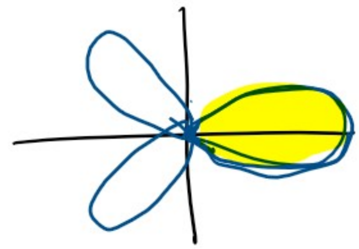
$$= \left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{4} + \frac{\sin\left(\frac{\pi}{2}\right)}{2}\right) - \left(0 + \cancel{\frac{\sin(0)}{2}}^{\,0}\right)$$

$$= \frac{\pi}{4} + \frac{1}{2} \quad \boxed{A}$$

$$A = \int^{\theta_2} \int^{f(\theta)} 1 \cdot r\,dr\,d\theta = \frac{1}{2} \int^{\theta_2} r^2\,d\theta$$

$$A = \int_{\theta_1}^{\theta_2}\int_0^{f(\theta)} 1 \cdot r\,dr\,d\theta = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta$$





$r = \cos(3\theta)$

$0 = \cos(3\theta)$

$\frac{\pi}{2} = 3\theta$

$\frac{\pi}{6} = \theta$