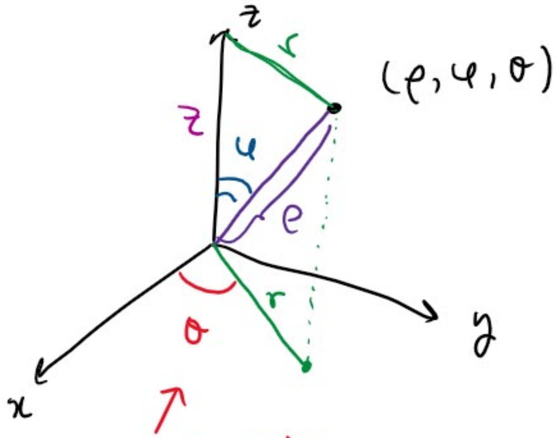
Lesson 23 - 10/20/2023 - 4:30pm

16.5: Triple Integrals in Spherical Coordinates



same θ as in polar

ρ = distance from origin in $\mathbb{R}^3$

φ = angle from $+z$-axis
$0 \le \varphi \le \pi$

θ = angle from $+x$-axis in the xy-plane
$0 \le \theta \le 2\pi$

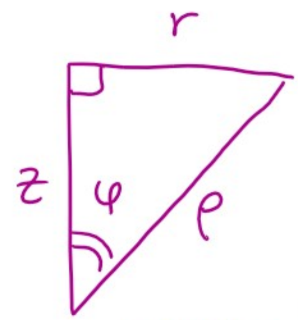
Conversions:

first convert to cylindrical

$r = \rho \sin\varphi$

$\theta = \theta$

$z = \rho\cos\varphi$



$\sin\varphi = \frac{r}{\rho}$ $\quad$ $\cos\varphi = \frac{z}{\rho}$

convert to Cartesian

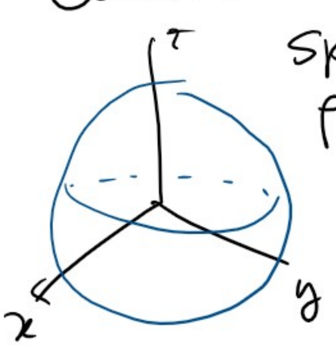
$x = r\cos\theta = \rho\sin\varphi\cos\theta$

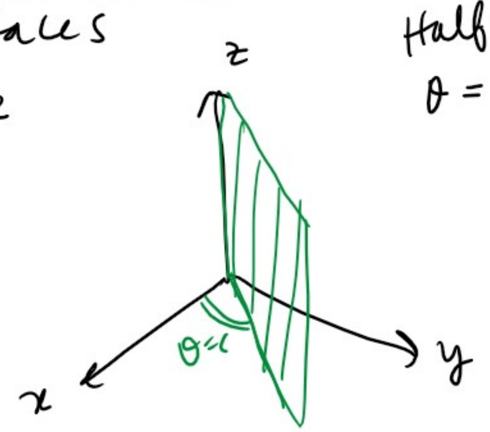
$y = r\sin\theta = \rho\sin\varphi\sin\theta$

$z = z \qquad = \rho\cos\varphi$

Common Surfaces

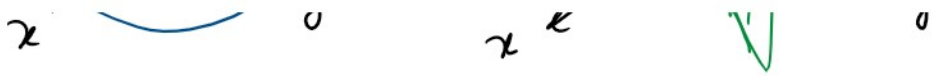
Sphere
$\rho = c$



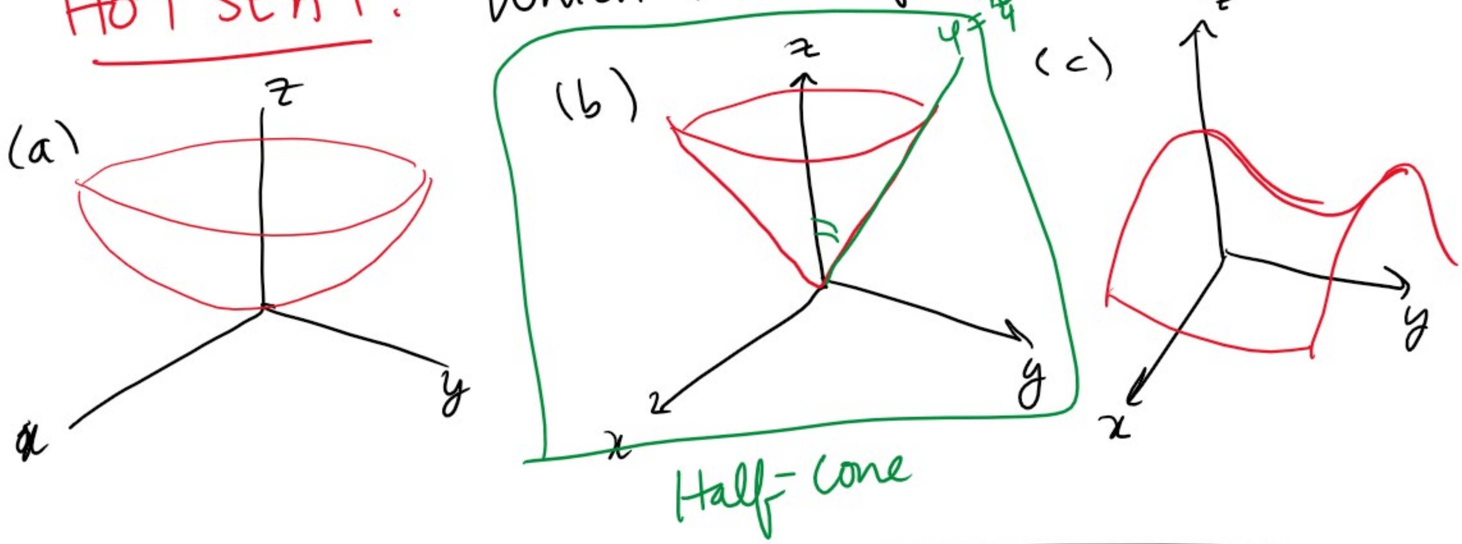


Half-Plane
$\theta = c$

... plot of $\varphi = \frac{\pi}{4}$



HOT SEAT: Which is the plot of $\varphi = \frac{\pi}{4}$



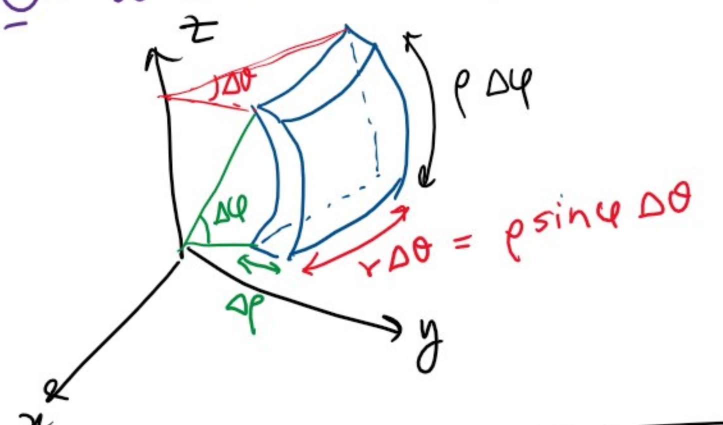
Fubini's Theorem in Spherical Coords:

"Spherical Box"

$$B = \{(\rho, \varphi, \theta) : a \le \rho \le b,\ \alpha \le \varphi \le \beta,\ \gamma \le \theta \le \psi\}$$

$$\iiint_B f(\rho, \varphi, \theta)\, dV = \int_{\theta=\gamma}^{\theta=\psi} \int_{\varphi=\alpha}^{\varphi=\beta} \int_{\rho=a}^{\rho=b} f(\rho, \varphi, \theta)\, \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

Q: Where does $\rho^2 \sin\varphi$ come from?

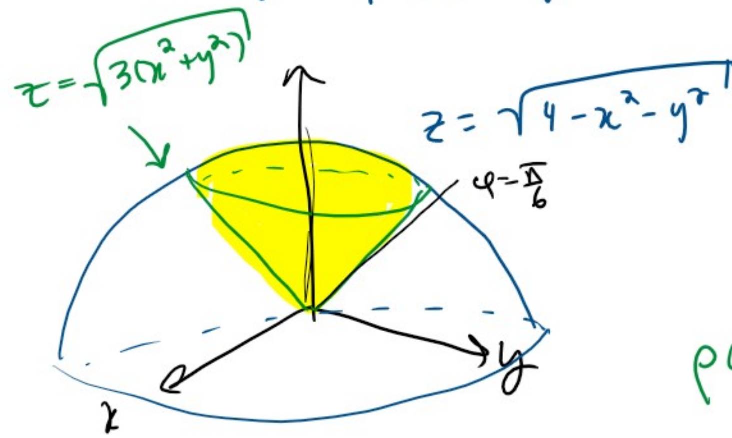


$$\Delta V = (\Delta\rho)(\rho\Delta\varphi)(\rho \sin\varphi \Delta\theta)$$

$$= (\rho^2 \sin\varphi)\, \Delta\rho\, \Delta\varphi\, \Delta\theta$$

Example: Set up the integral to find the volume of the region bounded by the cone

Example: Set up
volume of the region bounded by the cone $z = \sqrt{3(x^2+y^2)}$ and the hemisphere $z = \sqrt{4-x^2-y^2}$



sphere: $\rho = 2$

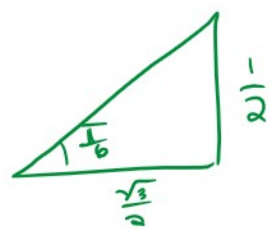
cone: $z = \sqrt{3(x^2+y^2)}$

$$\rho\cos\varphi = \sqrt{3(\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta)}$$

$$= \sqrt{3\rho^2\sin^2\varphi(\underbrace{\cos^2\theta + \sin^2\theta}_{1})}$$

$$\cancel{\rho}\cos\varphi = \sqrt{3}\,\cancel{\rho}\sin\varphi$$

$$\frac{1}{\sqrt{3}} = \tan\varphi \longrightarrow \boxed{\text{cone } \varphi = \frac{\pi}{6}}$$



$0 \le \rho \le 2 \qquad 0 \le \theta \le 2\pi$

$0 \le \varphi \le \frac{\pi}{6}$

$$\boxed{\text{Volume} = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^2 \rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta}$$

POLL 2: sphere of radius 2 centered @ (0,0,2)

$$(x-0)^2 + (y-0)^2 + (z-2)^2 = 2^2$$

$$\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta + (\rho\cos\varphi - 2)^2 = 4$$

$$\rho^2\sin^2\varphi[\underbrace{\cos^2\theta + \sin^2\theta}_{1}] + (\rho^2\cos^2\varphi - 4\rho\cos\varphi + \cancel{4}) = \cancel{4}$$

$$\rho^2(\underbrace{\sin^2\varphi + \cos^2\varphi}_{1}) - 4\rho\cos\varphi = 0$$

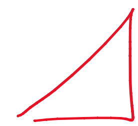
$$\rho^2 \cancel{(\sin^2\varphi + \cos^2\varphi)} - 4\rho\cos\varphi = 0$$

$$\rho^2 = 4\rho\cos\varphi$$

$$\boxed{\rho = 4\cos\varphi}$$

Two spheres intersect:

$$\rho = 2 \qquad \rho = 4\cos\varphi$$



$$2 = 4\cos\varphi$$

$$\cos\varphi = \frac{1}{2} \Rightarrow \boxed{\varphi = \frac{\pi}{3}}$$

$$0 \le \varphi \le \frac{\pi}{3}$$