Lesson 24 - 10/23/2023 - 4:30pm

16.6: Integrals for Mass Calculations

$z = \sqrt{x^2+y^2} \rightarrow$ cylindrical

$z^2 = x^2+y^2$

$\cancel{\rho^2}\cos^2\phi = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta$

$= \cancel{\rho^2}\sin^2\phi\,[\underbrace{\cos^2\theta+\sin^2\theta}_{1}]$

$\cos^2\phi = \sin^2\phi$

$1 = \tan^2\phi \quad \Rightarrow \quad \phi = \frac{\pi}{4}$

So far, we find the mass m of a region given density ρ

1D: $m = \int_a^b \rho(x)\,dx$

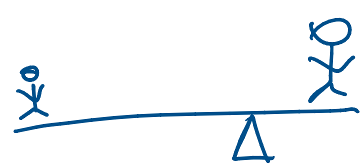
2D: $m = \iint_D \rho(x,y)\,dA$

3D: $m = \iiint_D \rho(x,y,z)\,dV$

Def: The center of mass is the point at which an object balances

Ex: See saw

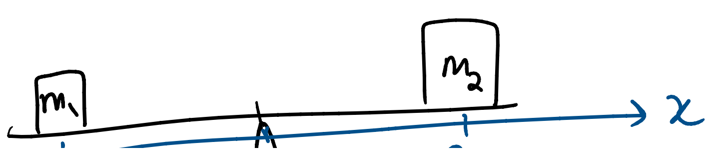
two kids of same size balances in middle

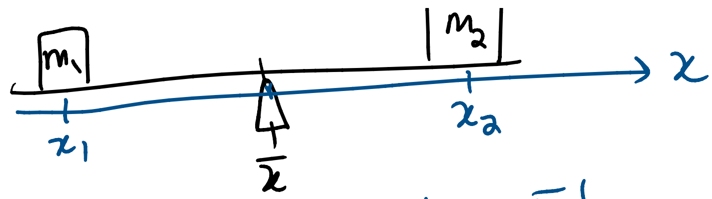


kids of different sizes balance off center

We need to find the mass moment

mass moment = mass $\cdot$ (distance from axis of rotation)





mass moments : $m_1 |x_1 - \bar{x}|$

$m_2 |x_2 - \bar{x}|$

The see saw balances when the moments are equal

In 1D:

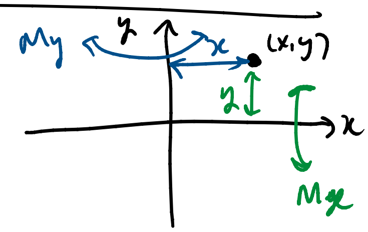
$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} \longleftarrow \text{discrete masses}$$

mass moment M

$$\bar{x} = \frac{\int_a^b x\,\rho(x)\,dx}{\int_a^b \rho(x)\,dx}$$

continous density $\rho(x)$ on $a \le x \le b$

In 2D | center of mass $(\bar{x}, \bar{y})$



Mass moment

M_x – moment about the x-axis

$= \text{mass} \cdot \underbrace{(\text{distance to the } x\text{-axis})}_{y}$

$$= \iint_D y \cdot \rho(x,y)\,dA$$

M_y = moment about the y-axis

$$= \iint_D x \cdot \rho(x,y)\,dA$$

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_D x \cdot \rho(x,y)\,dA}{\iint \rho(x,y)\,dA}$$

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_D \cdots}{\iint_D \rho(x,y)\,dA}$$

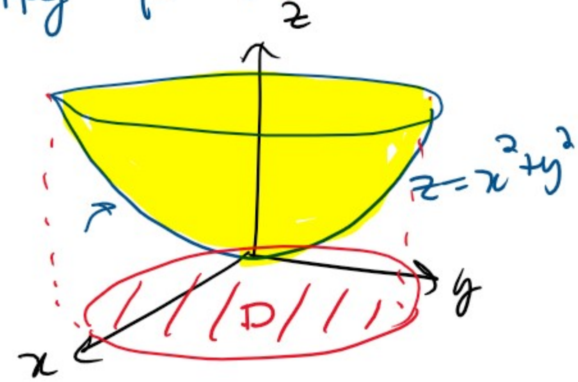
$$\bar{y} = \frac{M_x}{m} = \frac{\iint_D y \cdot \rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}$$

3D

$$m = \iiint_D \rho(x,y,z)\,dV \qquad \bar{x} = \frac{1}{m}\iiint_D x\,\rho(x,y,z)\,dV$$

$$\bar{y} = \frac{1}{m}\iiint_D y\,\rho(x,y,z)\,dV \qquad \bar{z} = \frac{1}{m}\iiint_D z\,\rho(x,y,z)\,dV$$

Example: Find the x-coordinate of the center of mass of the solid elliptic parabola $z = x^2+y^2$ between $z=0$ and $z=4$ whose density $\rho = 1$.



convert to cylindrical

$z = r^2$

@ $z=4$ $\quad 4 = r^2 \longrightarrow r = 2$

$0 \le \theta \le 2\pi$

$r^2 \le z \le 4$

$$m = \int_0^{2\pi}\int_0^2\int_{r^2}^4 1 \cdot r\,dz\,dr\,d\theta$$

$x = r\cos\theta$

$$\bar{x} = \frac{1}{m}\int_0^{2\pi}\int_0^2\int_{r^2}^4 x \cdot 1 \cdot r\,dz\,dr\,d\theta$$