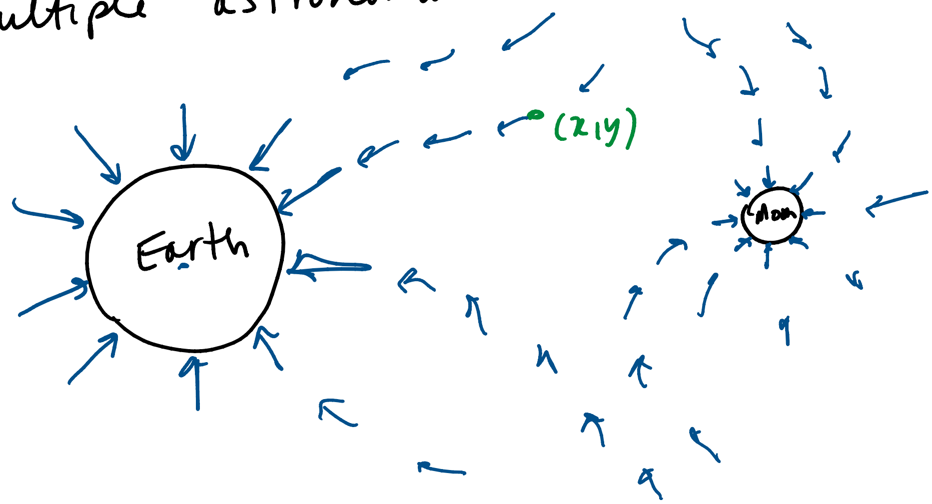
Lesson 25 - 10/25/2023 - 4:30pm

17.1: Vector Fields

Q: How can we model the gravitational force exerted by multiple astronomical objects?



Def: A vector field in $\mathbb{R}^2$ is an assignment of a 2-dimensional vector $\vec{F}(x,y)$ to each point (x,y) in D in $\mathbb{R}^2$

$\boxed{\text{In 3D}}$ $\vec{F}(x,y,z) \longrightarrow$ 3d vector in $\mathbb{R}^3$

A scalar field assigns a scalar at each point

$(x,y) \longrightarrow f(x,y)$

Ex: $f(x,y) = x+y$

$\boxed{\text{In 2D}}$ $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$

$= P(x,y)\hat{i} + Q(x,y)\hat{j}$

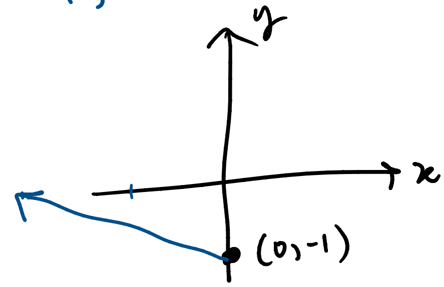
Example: $\vec{F}(x,y) = \langle 2y^2 + x - 4, \cos(x) \rangle$

at $(x,y) = (0,-1)$

$\vec{F} = \langle 2(-1)^2 + 0 - 4, \cos(0) \rangle = \langle -2, 1 \rangle$

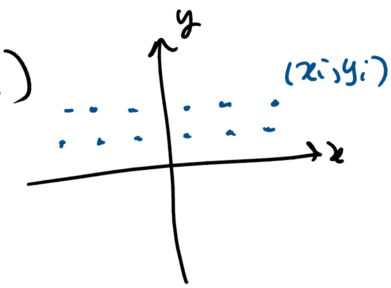
at $(x,y) = (0,-1)$

$\vec{F}(0,-1) = \langle 2(-1)^2 + 0 - 4,\ \cos(0) \rangle = \langle -2, 1 \rangle$



★ Draw a Vector Field:

1. choose a grid of points (x_i, y_i)
2. For each (x_i, y_i), draw the vector $\vec{F}(x_i, y_i)$ originating at (x_i, y_i)



$F(x,y) = \langle x, y \rangle$ is called a <u>radial field</u> — all vectors point either directly toward or directly away from the origin.

↳ magnitude only depends on the distance from the origin → i.e. $r = \sqrt{x^2+y^2}$

Ex: $\vec{F} = \langle x, y \rangle$

$|\vec{F}| = |\langle x, y \rangle| = \sqrt{x^2+y^2} = r$

<u>Ex</u>: radial fields: $\langle -2x, -2y \rangle$, $\langle \frac{5}{7}x, \frac{5}{7}y \rangle$

<u>Ex</u>: $\vec{F}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle = \left\langle \frac{x}{r}, \frac{y}{r} \right\rangle$

$= \underbrace{\frac{1}{r}}_{\sim \text{length}} \underbrace{\langle x, y \rangle}_{\text{direction}}$

gravitational field

~ length

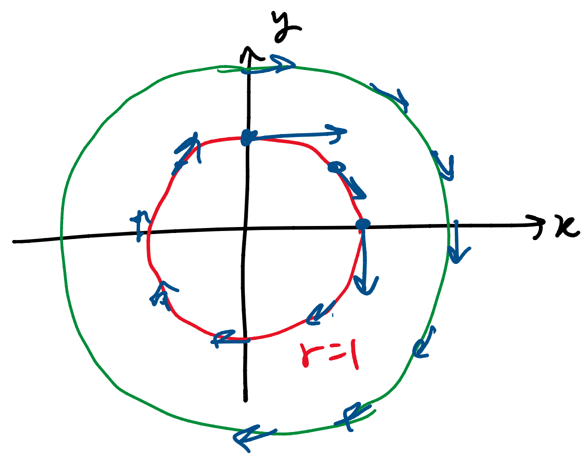
Ex: $\vec{F}_G = -Gm_1m_2 \left\langle \frac{x}{r^3}, \frac{y}{r^3} \right\rangle$ ← gravitational field

$= -\frac{Gm_1m_2}{r^2} \underbrace{\left\langle \frac{x}{r}, \frac{y}{r} \right\rangle}_{\text{unit vector}}$

A rotational field – the vector at point (x,y) is tangent to a circle with radius $r = \sqrt{x^2+y^2}$)

Ex: $\vec{F} = \langle y, -x \rangle$

(x,y)	$\vec{F}$
$(1,0)$	$\langle 0,-1 \rangle$
$(0,1)$	$\langle 1, 0 \rangle$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$



Def: A vector field $\vec{F}$ is a unit vector field if the magnitude of each vector is one

Ex: $\vec{F}(x,y) = \left\langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle = \left\langle \frac{-y}{r}, \frac{x}{r} \right\rangle$

$\vec{G}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle = \left\langle \frac{x}{r}, \frac{y}{r} \right\rangle = \frac{\vec{r}}{r}$

$\vec{r} = \langle x, y \rangle \quad r = |\vec{r}|$

Gradient Fields:

→ gravitational fields

→ electric fields associated with static charge

→ gravitational fields
→ electric fields associated with static charge

Def: A vector field $\vec{F}$ is a gradient field if there exists a scalar function φ such that

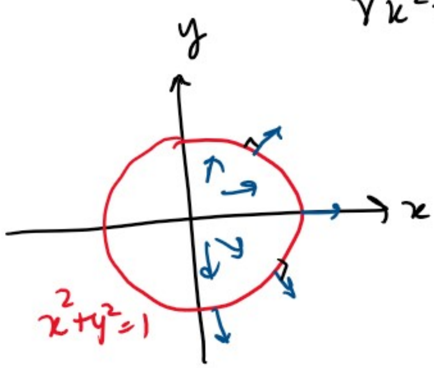
$$\vec{F} = \vec{\nabla}\varphi$$

φ is called the potential function

Ex: let $\varphi = x^2y^2$

$$\vec{F} = \nabla\varphi = \langle 2xy^2, 2x^2y \rangle$$

Example: Let C be the circle $x^2+y^2=1$. Show that at each point in C, the vector field $\vec{F} = \dfrac{\langle x, y\rangle}{\sqrt{x^2+y^2}}$ is orthogonal to the line tangent to C at that point



parameterize C

$$\vec{r}(t) = \langle \underset{x}{\cos(t)}, \underset{y}{\sin(t)} \rangle$$

tangent vector:

$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$$

vector field $\vec{F}$ on pts in C

$$\vec{F} = \frac{\langle x, y\rangle}{\sqrt{x^2+y^2}} = \frac{\langle \cos(t), \sin(t)\rangle}{\sqrt{\cos^2 t + \sin^2(t)}} = \langle \cos(t), \sin(t) \rangle$$

Want: $\vec{F} \perp \vec{r}\,'$

$$\vec{F} \cdot \vec{r}\,' \overset{?}{=} 0$$

$$\langle \ldots(t), \ldots \rangle \cdot \langle \ldots(t), \cos(t) \rangle \overset{?}{=} 0$$

$$\vec{F} \cdot \vec{r}\,' \doteq 0$$

$$\langle \cos(t), \sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \stackrel{?}{=} 0$$

$$-\sin(t)\cos(t) + \sin(t)\cos(t) = 0 \quad \checkmark$$