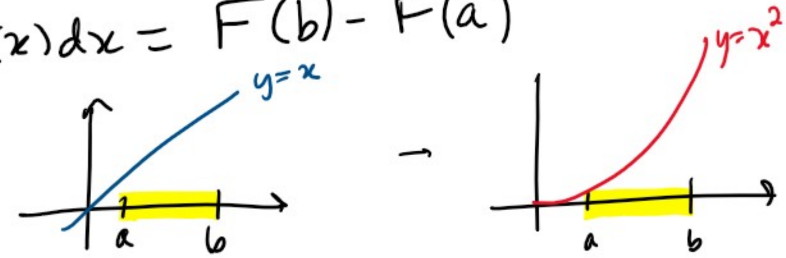
Lesson 26 - 10/27/2023 - 4:30pm

17.2: Line Integrals - Part 1

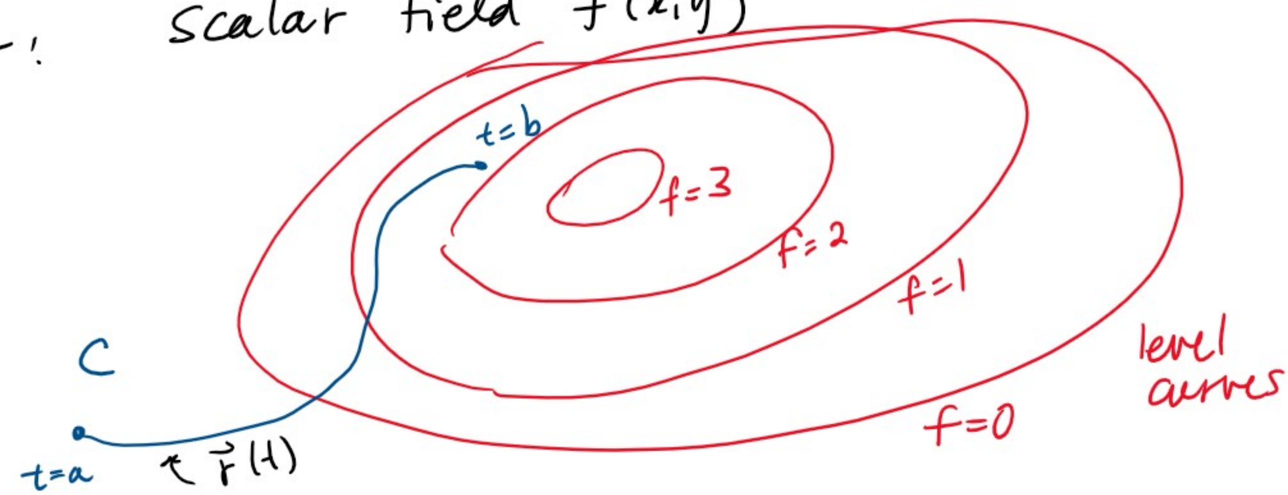
Calc 1: $\int_a^b f(x)\,dx = F(b) - F(a)$



Lesson 6: Given $\vec{r}\,'(t) = \vec{v}(t)$ vector valued function

$$\int_a^b \vec{r}\,'(t)\,dt = \vec{r}(b) - \vec{r}(a)$$ ← another vector-valued function

Today: scalar field $f(x,y)$



Want to integrate $f(x,y)$ along the 1D curve C

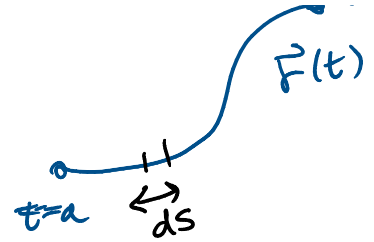
C - parameterize $\vec{r}(t)$ $a \le t \le b$

Line Integral $$\int_C f(x,y)\,ds$$ ← length of a small segment of curve

curve → C

We get ds from arclength

$$L = \int_a^b \underbrace{|\vec{r}\,'(t)|\,dt}$$



$$L = \int_a \underbrace{|\vec{r}\,'(t)|\, dt}_{ds}$$

Line Integral

$$\int_C f(x,y)\, ds = \int_a^b f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$$

where parameterize C by $\vec{r}(t)$ for $a \le t \le b$

<u>Example</u> Arclength: $f(x,y) = 1$

then $\int_C 1 \cdot ds = \int_a^b |\vec{r}\,'(t)|\, dt = L$ is arclength

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To evaluate a line integral $\int_C f(x,y)\, ds$

1. Parameterize C : $\vec{r}(t)$ for $a \le t \le b$
2. Find $|\vec{r}\,'(t)|$
3. Rewrite $\int_a^b f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$ Calc I integral

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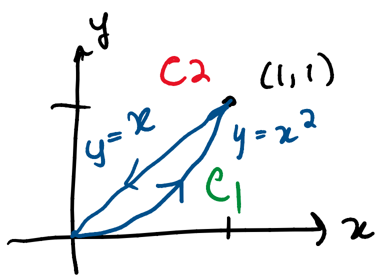
Common Parameterizations

Curve C	$\vec{r}(t)$
Circle of radius a centered at $(0,0)$	$\vec{r}(t) = \langle a\cos(t), a\sin(t) \rangle$

Circle of radius a centered at $(0,0)$	$r(t) = \langle a\cos(t), a \ldots$
Ellipse with lengths a, b centered at $(0,0)$	$\vec{r}(t) = \langle a\cos(t), b\sin(t) \rangle$
line through point $P = (x_0, y_0, z_0)$ and with direction vector $\vec{v}$	$\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t\vec{v}$ $= \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle$
parabola $y = x^2$	$\vec{r}(t) = \langle t, t^2 \rangle$
helix	$\vec{r}(t) = \langle a\cos(t), b\sin(t), ct \rangle$

<u>Ex:</u> $\int_C (x + \sqrt{y})\, ds$

where C: ① from $(0,0)$ to $(1,1)$ along $y = x^2$
② from $(1,1)$ to $(0,0)$ along $y = x$



← closed curve (end up where you started)

$$\int_C (x + \sqrt{y})\, ds = \int_{C_1} (x + \sqrt{y})\, ds + \int_{C_2} (x + \sqrt{y})\, ds$$

$\boxed{C_1}$ $\vec{r}(t) = \langle t, t^2 \rangle$ from $0 \le t \le 1$

$|\vec{r}\,'(t)| = |\langle 1, 2t \rangle| = \sqrt{1 + 4t^2}$

$\boxed{C_2}$ direction vector $\vec{v} = \langle 0-1, 0-1 \rangle = \langle -1, -1 \rangle$

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starting point $P = (1,1)$

$\vec{r}(\hat{t}) = \langle 1, 1 \rangle + \hat{t}\langle -1, -1 \rangle = \langle 1-\hat{t}, 1-\hat{t} \rangle$

@ $\hat{t}=0$ $\vec{r}(0) = \langle 1, 1 \rangle$ $\qquad 0 \le \hat{t} \le 1$

@ $\hat{t}=1$ $\vec{r}(1) = \langle 0, 0 \rangle$

$$\underbrace{\int_0^1 \left(t + \sqrt{t^2}\right)\sqrt{1+4t^2}\, dt}_{C_1} + \underbrace{\int_0^1 \left((1-\hat{t}) + \sqrt{1-\hat{t}}\right)\sqrt{2}\, d\hat{t}}_{C_2}$$