Lesson 27 - 10/30/2023 - 4:30pm

17.2: Line Integrals - Part 2

Last Class: $\int_C f(x,y)\,ds$

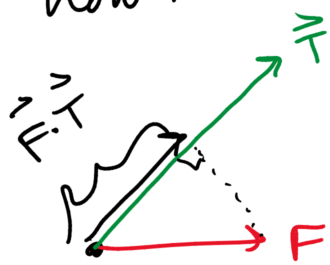
scalar line integral because $f(x,y)$ is scalar field

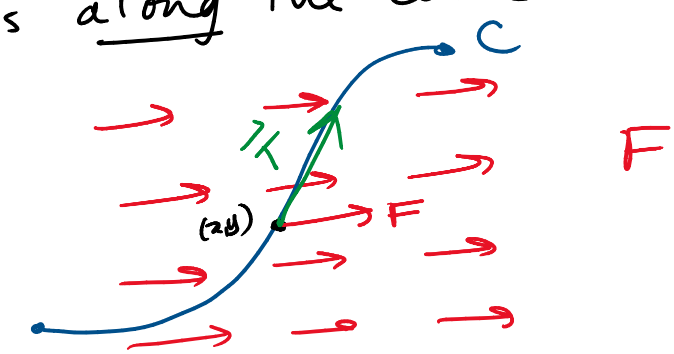
Today: vector line integrals

$$\int_C \vec{F}(x,y)\cdot\vec{T}\,ds$$

here $\vec{F}$ is a vector field

how much $\vec{F}$ points along the curve C





$\vec{F}\cdot\vec{T}$ = the proportion of $\vec{F}$ in the direction $\vec{T}$

Recall, $\vec{T}$ is the unit tangent vector

$$\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \leftarrow \text{unit vector}$$

Recall from last class

$$ds = |\vec{r}\,'(t)|\,dt$$

Assume $\vec{r}(t)$, $a \le t \le b$

$$\int \vec{F}\cdot\vec{T}\,ds = \int^b \vec{F}(\vec{r}(t))\cdot\left(\frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}}\right)\underbrace{\cancel{|\vec{r}\,'(t)|}\,dt}_{ds}$$

$$\int_C \vec F\cdot\vec T\,ds = \int_a \vec F(\vec r(t))\cdot \underbrace{\left(\cancel{|\vec r'(t)|}\right)}_{\vec T}\ \underbrace{\qquad}_{ds}$$

$$\boxed{\int_C \vec F\cdot\vec T\,ds = \int_a^b \vec F(\vec r(t))\cdot\vec r'(t)\,dt}$$

where $\vec r(t)$ parameterizes C for $a \le t \le b$

$$\int_C f(x,y)\,ds = \int_a^b f(\vec r(t))\,|\vec r'(t)|\,dt$$ scalar line integral

<u>NOTE</u>: Sometimes its written

$$\int_C \vec F\cdot\vec T\,ds = \int_C \vec F\cdot d\vec r \qquad d\vec r = \vec r'(t)\,dt$$

or, if $\vec F = \langle P(x,y),\ Q(x,y)\rangle$
$\vec r(t) = \langle x(t),\ y(t)\rangle$

$$\int_C \vec F\cdot\vec T\,ds = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

$dx = x'(t)\,dt$
$dy = y'(t)\,dt$

Work Work = Force x distance

Work done along curve C due to force $\vec F$

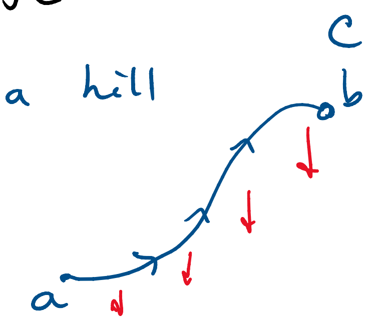
$$W = \int_C \vec F\cdot\vec T\,ds = \int_C \vec F\cdot d\vec r$$

$$W = \int_C F \cdot T \, ds = \int_C F \cdot dr$$

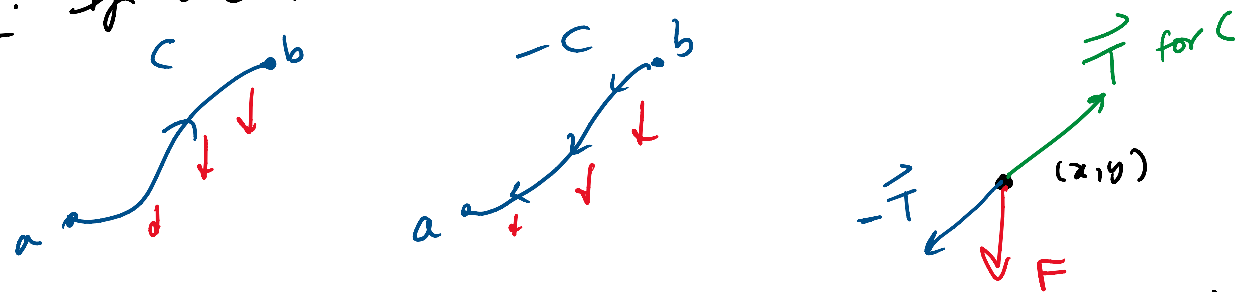
Ex: Work needed to hike up a hill

$\vec{F} = \vec{F_g}$ - gravity

C - path



NOTE: If we reverse the orientation of C

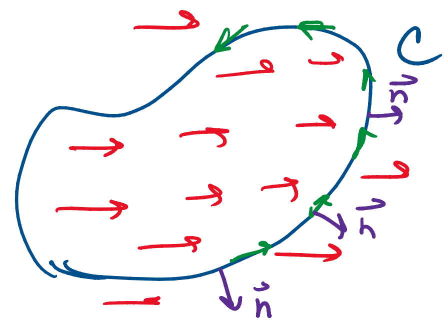


$$\int_{-C} \vec{F} \cdot d\vec{r} = - \int_C \vec{F} \cdot d\vec{r} \qquad -\vec{F} \cdot \vec{T} = \vec{F} \cdot (-\vec{T})$$

Flux: $\vec{F}$ - velocity of a fluid

C - membrane or walls of a cell

Q: How much of the flow $\vec{F}$ is leaving the cell C?



$\int_C \vec{F} \cdot \vec{T} \, ds$

How much across curve C

Need is the normal vector $\vec{n}$

$\vec{n}$ is orthogonal to C at that point

$\vec{n} \cdot \vec{T} = 0 \qquad \vec{n} \perp \vec{T}$

Flux:

$$\int_C \vec{F}\cdot\vec{N}\,ds = \int_C \vec{F}\cdot\underbrace{\left(\frac{\vec{n}}{|\vec{n}|}\right)}_{\text{unit normal vector}}\underbrace{|\vec{r}\,'|\,dt}_{ds}$$

$\vec{r}(t)$ parameterizes C,

$\vec{r}(t) = \langle x(t), y(t)\rangle$ $\qquad \vec{n} \perp \vec{r}\,' = \vec{T}$

$\vec{r}\,'(t) = \langle x'(t), y'(t)\rangle$

$\vec{n}(t) = \langle y'(t), -x'(t)\rangle$ nomal vector

$$\vec{N}(t) = \frac{\vec{n}(t)}{|\vec{n}(t)|} = \frac{\vec{n}(t)}{\sqrt{(y')^2+(-x')^2}} = \frac{\vec{n}(t)}{|\vec{r}\,'(t)|}$$

Flux:

$$\int_C \vec{F}\cdot\vec{N}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\left(\frac{\vec{n}(t)}{\cancel{|\vec{r}\,'(t)|}}\right)\cancel{|\vec{r}\,'(t)|}\,dt$$

$$\boxed{\int_C \vec{F}\cdot\vec{N}\,ds = \int_a^b \vec{F}(r(t))\cdot\vec{n}(t)\,dt}$$

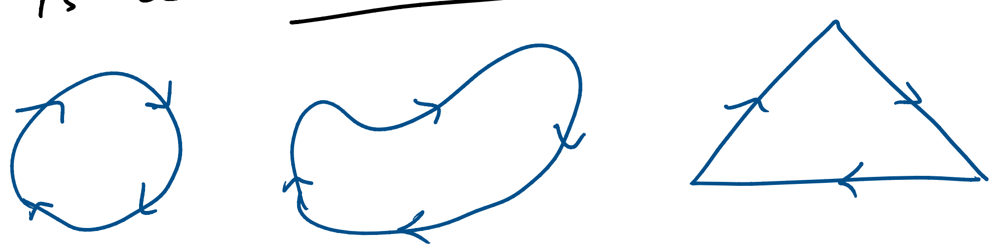
In 2D:

$$\int_C \vec{F}\cdot\vec{N}\,ds \overset{?}{=} \int_C |\vec{F}\times\vec{T}|\,ds$$

Circulation

If C is a closed curve

Ex:



If C is a closed curve, we call

If C is a closed curve, we call $\int_C \vec{F}\cdot\vec{T}\,ds$ the circulation

and write $\int_C \vec{F}\cdot\vec{T}\,ds = \oint_C \vec{F}\cdot\vec{T}\,ds$