Lesson 28 - 11/1/2023 - 4:30pm

17.3: Conservative Vector Fields and the Fundamental Theorem of Line Integrals

Calc 1 Fundamental Thm of calculus

If $f(x)$ has antiderivative $F(x)$ then

$$\int_a^b f(x)\,dx = F(b) - F(c)$$

Calc 3 Think of $\vec{\nabla}$ (gradient) as a derivative

Curve C has parameterization

$\vec{r}(t) = \langle x(t), y(t) \rangle$ for $a \le t \le b$

Let $f(x,y)$ be a scalar field whose derivatives that exist and are continuous on C

Then $\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \vec{F}$ is a gradient vector field

$$\int_C \vec{\nabla} f \cdot d\vec{r} = \int_a^b \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_a^b \left\langle \frac{\partial f}{\partial x}\Big|_{\vec{r}(t)}, \frac{\partial f}{\partial y}\Big|_{r(t)} \right\rangle \cdot \langle x'(t), y'(t) \rangle\,dt$$

$$= \int_a^b \Bigg(\underbrace{\frac{\partial f}{\partial x}\Big|_{\vec{r}(t)} x'(t) + \frac{\partial f}{\partial y}\Big|_{\vec{r}(t)} y'(t)}_{\text{Chain Rule}} \Bigg)\,dt$$

$$\frac{d}{dt}\left(f(\vec{r}(t)) \right)$$

$$= \int_a^b \frac{d}{dt}\left(f(\vec{r}(t))\right)dt = \Big[f(\vec{r}(t)) \Big]_{t=a}^{t=b}$$

$$\int \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

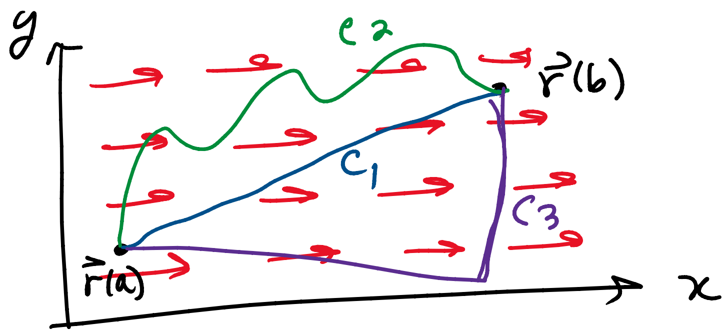
$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Fundamental Thm of Line Integrals

C - parameterized by $\vec{r}(t)$ $a \le t \le b$

$f = x$

$\vec{F} = \nabla f = \langle 1, 0 \rangle$



$$\int_{C_1} \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = \int_{C_2} \vec{\nabla} f \cdot d\vec{r} = \int_{C_3} \vec{\nabla} f \cdot d\vec{r}$$

We say $\vec{F}$ is <u>independent of path</u> if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot dr$ as long as C_1 and C_2 have the same endpoints

<u>Thm:</u> If $\vec{F}$ is a conservative field (i.e. there exist f so that $\vec{F} = \vec{\nabla} f$) then $\vec{F}$ is independent of path.

<u>Q</u>: How do you know if $\vec{F}$ is conservative?

<u>A</u>: Assume $\vec{F} = \langle P(x,y), Q(x,y) \rangle = \vec{\nabla} f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

$\frac{\partial}{\partial y}\left(P = \frac{\partial f}{\partial x}\right)$ $\qquad$ $\frac{\partial}{\partial x}\left(Q = \frac{\partial f}{\partial y}\right)$ ... because

$$\frac{\partial}{\partial y}\left(P = \frac{\partial f}{\partial x}\right) \qquad \frac{\partial}{\partial x}\left(Q = \frac{\partial f}{\partial y}\right)$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad = \quad \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

* because $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

$$\boxed{\vec{F}(x,y) = \langle P, Q \rangle \text{ is conservative if } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

Q: Given $\vec{F}$ how do you find potential f?

$$\vec{F} = \langle P(x,y), Q(x,y) \rangle = \vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$P = \frac{\partial f}{\partial x} \qquad Q = \frac{\partial f}{\partial y}$$

$$f \sim \int \frac{\partial f}{\partial x} dx = \int P \, dx$$

$$f \sim \int \frac{\partial f}{\partial y} dy = \int Q \, dy$$

Example: $\vec{F} = \langle 2xy + e^x, x^2 + \cos(y) \rangle$

find the potential f

$$f = \int \frac{\partial f}{\partial x} dx = \int (2xy + e^x) dx = x^2 y + e^x + \underbrace{a(y)}_{\text{function of integration}}$$

$$\approx \int \frac{\partial f}{\partial y} dy = \int (x^2 + \cos(y)) dy = x^2 y + \sin(y) + b(x)$$

To get $f(x,y)$, we need to match terms

$$f(x,y) = x^2 y + e^x + \sin(y) + C$$

$f(x,y) = x^{2}y + e \; + \sin(y) + C$

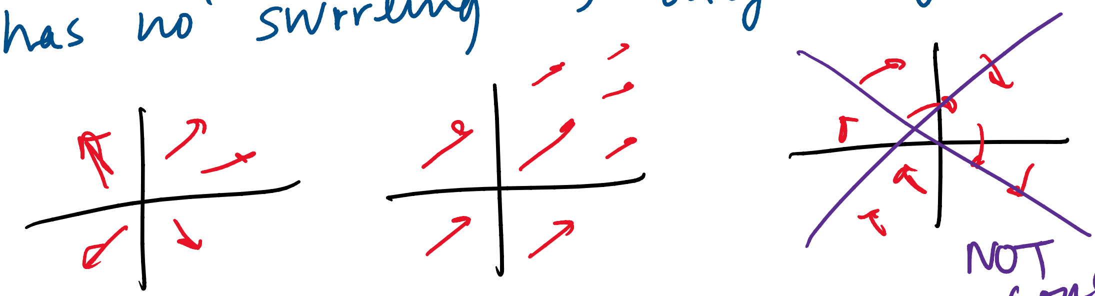
Q: What does a conservative vector field look like?

$$\vec{\nabla} \times \left(\vec{F} = \vec{\nabla} f \right)$$

$$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times (\vec{\nabla} f) = 0$$

Optional
ignore until we get to chapter on curl.

has no swirling $\rightarrow$ only straight lines



NOT conservative