Lesson 29 - 11/3/2023 - 4:30pm

17.4: Green's Theorem

Curl:

Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ is a 2D vector field

Think $\vec{F}(x,y) = \langle P, Q, 0 \rangle$

Define $\vec{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ "del"

The curl of $\vec{F}$:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix}$$

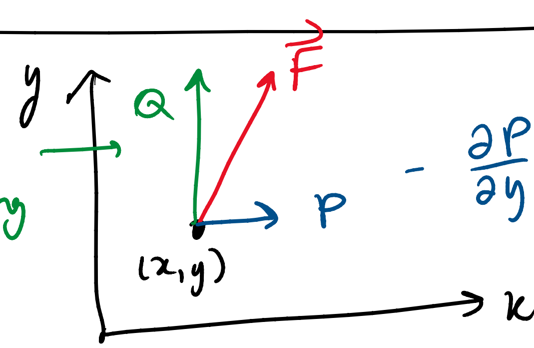
$P = P(x,y)$
$Q = Q(x,y)$

$$= \left(0 - \cancelto{0}{\tfrac{\partial}{\partial z} Q}\right)\hat{i} - \left(0 - \cancelto{0}{\tfrac{\partial}{\partial z} P}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

$$= \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

NOTE: In 2D, $\vec{F} = \langle P, Q \rangle$, then
$$\vec{\nabla} \times \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \quad \text{is scalar}$$

$\frac{\partial Q}{\partial x}$: how much $\uparrow$ is changing left/right

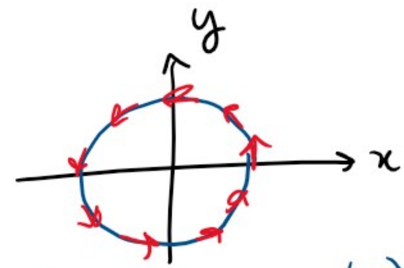


$- \frac{\partial P}{\partial y}$: how much $\rightarrow$ is changing up/down

$\vec{\nabla} \times \vec{F}$ measures how much a vector field circulates or rotates

circulates or rotates

Ex: $\vec{F} = \langle -y, x \rangle$ rotational field



$$\vec{\nabla} \times \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 1 - (-1) = 2$$

curl is positive everywhere, so Field $\vec{F}$ is rotational everywhere

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If $\vec{F} = \nabla \phi$ is conservative then the curl is zero everywhere. irrotational

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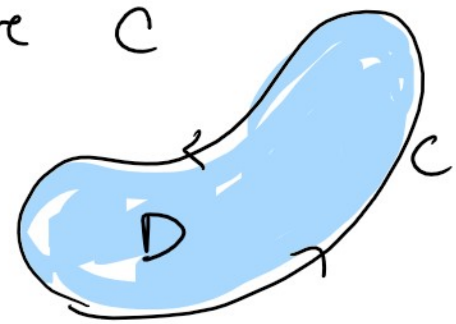
Assume we have a closed curve C

It encloses a region D

Then if $\vec{F} = \langle P, Q \rangle$



Green's Theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D (\vec{\nabla} \times \vec{F})\, dA$$

Why? If $\vec{F} = \vec{\nabla}\phi$ is conservative

then $\oint_C \vec{F} \cdot d\vec{r} = \phi(b) - \phi(a) = 0$ $\quad a = b$

also $\vec{\nabla} \times \vec{F} = \vec{\nabla} \times \vec{\nabla}\phi = 0$ ✓

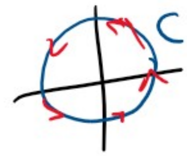
$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0$$

also $\nabla \times F$

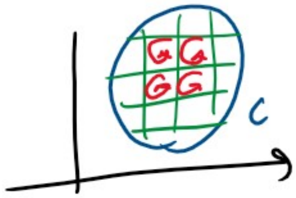
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0$$

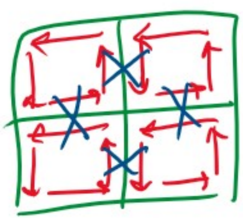
<u>More Generally:</u>

$\oint_C \vec{F}\cdot d\vec{r}$ – line integral that accumulates $\vec{F}$ along C

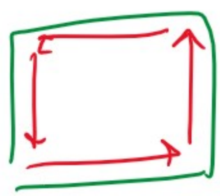


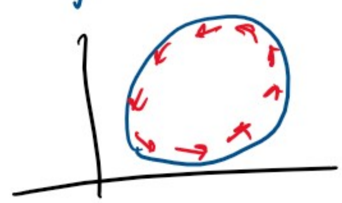
$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$ – accumulating rotation of $\vec{F}$ over D





rotation cancels out on the boundary of 2 neighboring boxes





$$\vec{\nabla} \times \vec{E} = \frac{\partial B}{\partial t}$$

Use Green's Thm to calculate areas:

<u>Ex</u>: Calculate the area of an ellipse $\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$

$$\text{area} = \iint_D 1 \cdot dA$$

Green's Thm: $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_C \vec{F}\cdot d\vec{r}$

Need: $\vec{F} = \langle P, Q\rangle$ whose $\vec{\nabla} \times \vec{F} = 1$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

Need: $\vec{F} = \langle P, Q \rangle$ whose $\nabla \times F = 1$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

$$\frac{\partial Q}{\partial x} = \frac{1}{2}, \qquad \frac{\partial P}{\partial y} = -\frac{1}{2}$$

$$\int \frac{\partial Q}{\partial x} = \int \frac{1}{2}\, dx \rightarrow Q = \frac{x}{2}$$

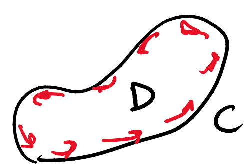
$$\int \frac{\partial P}{\partial y}\, dy = \int -\frac{1}{2}\, dy \rightarrow P = -\frac{y}{2}$$

$$\vec{F} = \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle$$

$$\text{area} = \iint_D dA = \oint_C \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle \cdot d\vec{r}$$

Green's Thm:

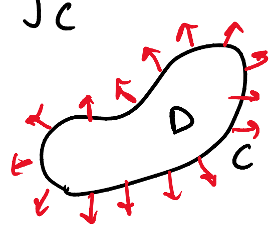
$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot \vec{T}\, ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



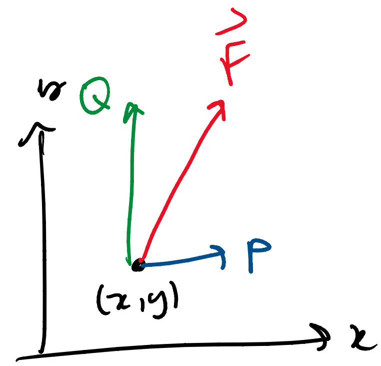
$$\oint_C \vec{F} \cdot \vec{N}\, ds = \iint_D \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)}_{\text{Divergence}} dA$$

Divergence

$\vec{\nabla} \cdot \vec{F}$



$\frac{\partial Q}{\partial y}$: measures how much ↑ is changing up/down



Think of divergence as how much $\vec{F}$ is expanding.

Ex: $\vec{F} = \langle -y, x \rangle$ rotational field

$\vec{\nabla} \cdot \vec{F} = ?$

Ex: $\vec{F} = \langle -y, x \rangle$

$$\vec{\nabla} \times \vec{F} = 2$$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$$

we say $\vec{F}$ is source-free

$\vec{F} = \langle x, y \rangle$ radial field, conservative

$$\vec{\nabla} \times \vec{F} = 0 \quad \text{irratational}$$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2.$$