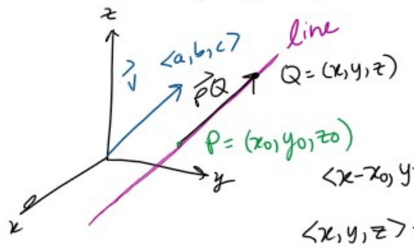
Lesson 2 - 8/23/2023 - 4:30pm

☆Lines & Planes in Space

Lines in 3D:

Q: Find the equation of the line parallel to vector $\vec{v} = \langle a, b, c\rangle$ and through point $P = (x_0, y_0, z_0)$



$\vec{PQ} \parallel \vec{v}$

$\vec{PQ} = t\vec{v}$ — t is a scalar

$\langle x - x_0, y - y_0, z - z_0\rangle = t\langle a, b, c\rangle$

$\underbrace{\langle x, y, z\rangle}_{\vec{r} \text{ dep. var}} - \underbrace{\langle x_0, y_0, z_0\rangle}_{\vec{r}_0} = \underbrace{t}_{\text{indep var (parameter)}} \underbrace{\langle a, b, c\rangle}_{\vec{v}}$

$$\boxed{\text{Vector equation of line} \quad \vec{r} = \vec{r}_0 + t\vec{v}}$$

direction vector

another way to write

$\langle x - x_0, y - y_0, z - z_0\rangle = t\vec{v} = \langle ta, tb, tc\rangle$

$$\begin{cases} x - x_0 = ta \\ y - y_0 = tb \\ z - z_0 = tc \end{cases} \rightarrow \quad \frac{x - x_0}{a} = t, \quad \frac{y - y_0}{b} = t, \quad \frac{z - z_0}{c} = t$$

$$\boxed{\text{Symmetric equation of line} \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

☆Relationships between lines:

Two lines can be:
1. equal
2. parallel
3. intersecting
4. skew

Def: Two lines that are not parallel and do not intersect are called skew

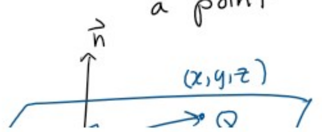
		Lines share a common point?	
		Yes	NO
Are Direction Vectors parallel?	Yes	Equal	Parallel
	NO	Intersecting	Skew

Equation of a Plane:

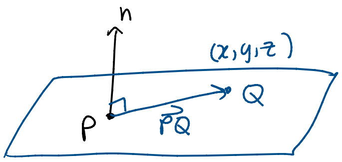
Given a normal vector $\vec{n} = \langle a, b, c\rangle$
a point $P = (x_0, y_0, z_0)$

$\vec{PQ} \cdot \vec{n} = 0$

$\vec{PQ} \perp \vec{n}$



$\langle x - x_0, y - y_0, z - z_0\rangle \cdot \langle a, b, c\rangle = 0$



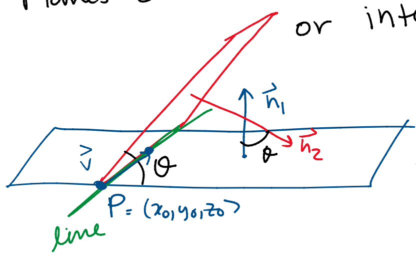
$\vec{PQ} \perp n$

$\langle x-x_0, y-y_0, z-z_0\rangle \cdot \langle a,b,c\rangle = 0$

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$
Equation of a plane

☆ Relationships Between Planes:

Planes can be: (equal)
parallel
or intersecting



angle θ between two planes

$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2|\cos\theta$

$\theta = \cos^{-1}\left(\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$

Q: What is the intersection between two planes $\rightarrow$ is a line

point P is in both planes

eqn of line of intersection

$\vec{r} = \langle x_0, y_0, z_0\rangle + t\,\vec{v}$

How to find $\vec{v}$?

$\vec{v}$ is in plane 1
$\vec{v}\cdot\vec{n}_1 = 0$

$\vec{v}$ is in plane 2
$\vec{v}\cdot\vec{n}_2 = 0$

$\vec{v} = \vec{n}_1 \times \vec{n}_2$

Example: Plane 1: $x+y+z=0$
Plane 2: $2x-y+z=0$

$a(x-\cancel{x_0}^{0}) + b(y-\cancel{y_0}^{0}) + c(z-\cancel{z_0}^{0}) = 0$

$ax+by+cz=0$

$\vec{n} = \langle a, b, c\rangle$

line of intersection: both go through origin: $\vec{r}_0 = \langle 0,0,0\rangle$

$\vec{n}_1 = \langle 1,1,1\rangle$ $\quad \vec{n}_2 = \langle 2,-1,1\rangle$

direction vector $\vec{v} = \vec{n}_1 \times \vec{n}_2$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 1 & 1 \\ -1 & 1\end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 1 \\ 2 & 1\end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ 2 & -1\end{vmatrix}$$

$$= \hat{i}(1\cdot 1 - (1)(-1)) - \hat{j}(1\cdot 1 - 2\cdot 1) + \hat{k}(1(-1) - 2\cdot 1)$$

$$= 2\hat{i} + \hat{j} + -3\hat{k} = \langle 2, 1, -3\rangle$$

line of intersection:

$l: \vec{r}_0 + t\,\vec{v} = \boxed{\langle 0,0,0\rangle + t\langle 2,1,-3\rangle}$

Plane: Plane 1: $x + y + z = 0$

Plane 2: $(x-1) + 2(y-3) + z = 0$

What is the intersection?

$x + 2y + z = 0 + 1 + 6 = 7$

Solve:

$$\begin{cases} x + y + z = 0 \\ x + 2y + z = 7 \end{cases}$$

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$0 - y + 0 = -7 \rightarrow y = 7$

$x + y + z = 0$

$x + 7 + z = 0 \rightarrow$

$z = t$

$z = t$

$x = -z - 7 = -7 - t$

$y = 7$

$x + 2y + z = 7$

equation of line of intersection in symmetric form

$\langle x, y, z \rangle = \langle -7, 7, 0 \rangle + t \langle -1, 0, 1 \rangle$

$2x + y + z = 0$

$-\; x + 2y + z = 7$

$x - y = 7$

$x = 7 - y$

$2(7 - y) + y + z = 0$

$14 - y + z = 0$

$z = y - 14$

$y = t$

$z = t - 14$

$x = 7 - t$

$$\boxed{\langle x, y, z \rangle = \langle 7, 0, -14 \rangle + t \langle -1, 1, 1 \rangle}$$

intercept

slope