Lesson 30 - 11/8/2023 - 4:30pm

17.5: Divergence and Curl

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$ "del operator"

calc 1 : $\frac{d}{dx}$

gradient: $\vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

input: f scalar field $\longmapsto$ output: vector field

divergence: $\vec{\nabla} \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

input: $\vec{F}$ vector field $\longmapsto$ output: scalar field

If $\vec{\nabla} \cdot \vec{F} = 0$ we say $\vec{F}$ is <u>source free</u>

curl: $\vec{\nabla} \times \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R \rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

curl of a 2D vector field $\vec{F} = \langle P, Q, 0 \rangle$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

input: $\vec{F}$ vector field $\longmapsto$ output: vector field

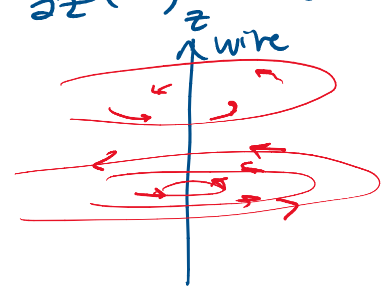
If $\vec{\nabla} \times \vec{F} = 0$, we say $\vec{F}$ is <u>irrotational</u>

's law: If $\vec{B}$ is a magnetic field,

Gauss's Law: If $\vec{B}$ is a magnetic field, then $\vec{\nabla} \cdot \vec{B} = 0$

Ex: Is $\vec{B} = \langle -y, x, 0 \rangle$ a magnetic field?

$$\vec{\nabla} \cdot \vec{B} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(0) = 0$$



Divergence tells us how much of the field is "flowing in" or "flowing out"

Green's Thm for Flux:

$$\text{Flux} = \oint_C \vec{F} \cdot \vec{N} \, ds = \iint_D \vec{\nabla} \cdot \vec{F} \, dA$$

Curl: $\vec{\nabla} \times \vec{F}$

imagine a flow $\vec{F}$, put a small paddle wheel in the flow, does it rotate

In 2D: $F = \langle P, Q, 0 \rangle$

$$\vec{\nabla} \times \vec{F} = \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle \quad = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x}$$

$\vec{\nabla} \times \vec{F} > 0$ — yes, rotates counter clockwise

$\vec{\nabla} \times \vec{F} = 0$ — NO, irrotational

$\vec{\nabla} \times \vec{F} < 0$ — yes, rotate clockwise

The general rotational vector field is $\vec{F} = \vec{a} \times \vec{r}$ where $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is called

The general rotat... $\vec{F} = \vec{a} \times \vec{r}$ where $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is called the axis of rotation and $\vec{r} = \langle x, y, z \rangle$

Ex: $\vec{a} = \langle 1, 2, 3 \rangle$

$$\vec{F} = \vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ x & y & z \end{vmatrix}$$

$$= \langle 2z - 3y, 3x - z, y - 2x \rangle$$

Q: Curl of a conservative vector field?

$$\vec{F} = \vec{\nabla}\phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle$$

$$\vec{\nabla} \times \vec{\nabla}\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \left\langle \frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}, \frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}, \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right\rangle$$

$$= \langle 0, 0, 0 \rangle$$ irrotational

$$\vec{\nabla} \cdot \vec{\nabla} f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$ Laplacian

$$\vec{\nabla} \cdot \vec{\nabla} f = \nabla^2 f$$
