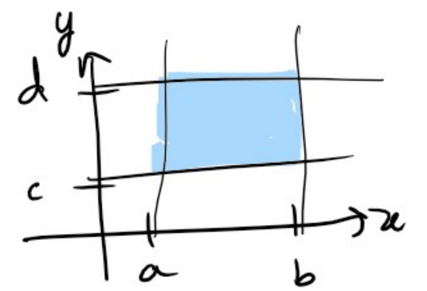
Lesson 31 - 11/10/2023 - 4:30pm

17.6: Surface Integrals (Part 1)

Line Integrals	Surface Integrals
$\int_C f(x,y,z)\,ds$ ← little s arclength	$\iint_S f(x,y,z)\,dS$ ← big S surface area
parameterize C $\vec{r}(t)$, for $a \le t \le b$	parameterize surface S $\vec{r}(u,v)$ for $a \le u \le b$, $c \le v \le d$
$\int_a^b f(\vec{r}(t))\underbrace{\lvert\vec{r}\,'(t)\rvert\,dt}_{ds}$	?

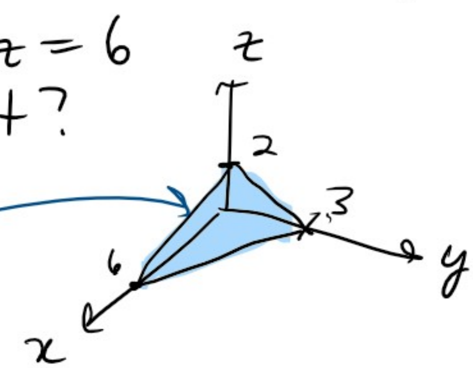
Double Integrals:
$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$



surface - flat rectangle $[a,b] \times [c,d]$ in the xy-plane $(z=0)$

Q: Want to integrate over the surface of the plane $x + 2y + 3z = 6$ in the first quadrant?

surface: this face of tetrahedron



parameterize surface

$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$

parameterize

$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$

One approach: Let $u = x$
$v = y$

$u + 2v + 3z = 6$

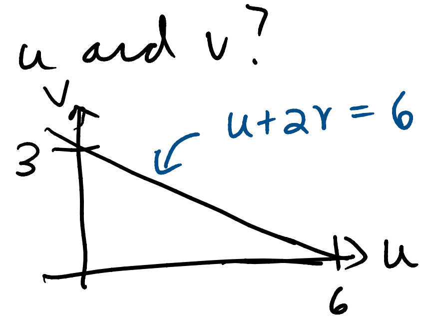
$z = \dfrac{6 - u - 2v}{3} = 2 - \frac{u}{3} - \frac{2v}{3}$

$\vec{r}(u,v) = \langle u, v, 2 - \frac{u}{3} - \frac{2v}{3} \rangle$

Q: What are the bounds on u and v?

$0 \le v \le 3 - \frac{u}{2}$

$0 \le u \le 6$



NOTE: Surface parameterizations are not unique

Q: Parameterize the surface of the sphere $x^2 + y^2 + z^2 = 4$ that lies in the first octant

spherical $\longrightarrow$ $\rho = 2$

$0 \le \underset{u}{\theta} \le \frac{\pi}{2}$

$0 \le \underset{v}{\phi} \le \frac{\pi}{2}$

$\vec{r}(u,v) = \langle 2\sin v \cos u,\ 2\sin v \sin u,\ 2\cos v \rangle$

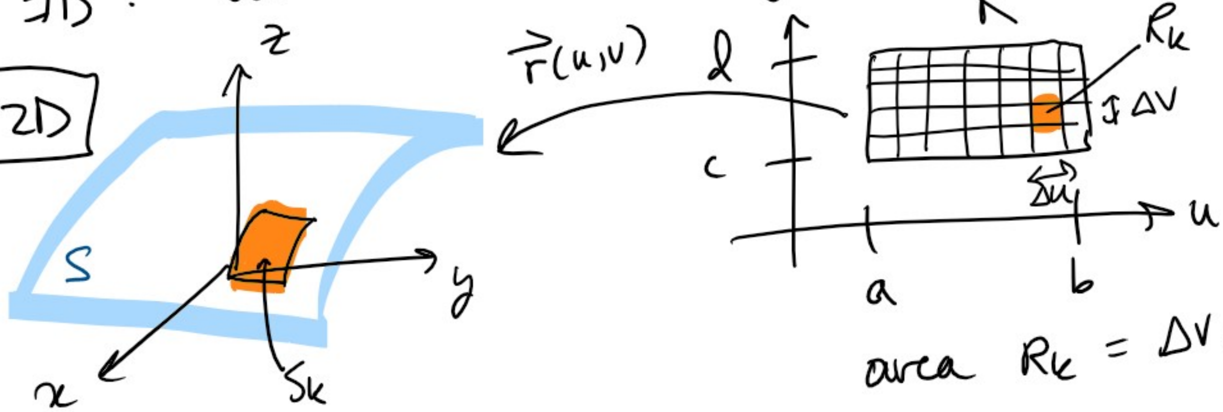
Surface Integrals:

$\iint f(x,y,z)\, dS$ ← What is dS

$$\iint_S f(x,y,z)\, dS$$

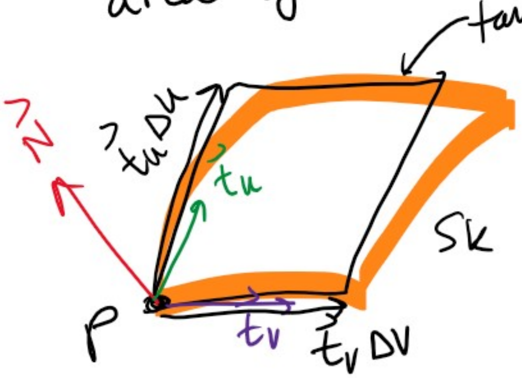
In 1D : ds arclength $ds = |\vec{r}\,'(t)|\, dt$

$\boxed{\text{In 2D}}$



area $R_k = \Delta v \Delta u$

area of S_k



tangent vectors

$$\vec{t}_u = \frac{\partial \vec{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

$$\vec{t}_v = \frac{\partial \vec{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

Area of S_k $\approx$ area of the parallelogram with sides $\vec{t}_u \Delta u$ and $\vec{t}_v \Delta v$

$$= |\vec{t}_u \Delta u \times \vec{t}_v \Delta v|$$

$$= |\vec{t}_u \times \vec{t}_v|\, \Delta u \Delta v$$

In the limit as $\Delta u, \Delta v \longrightarrow 0$

$$dS = |\vec{t}_u \times \vec{t}_v|\, du\, dv$$

Surface S is parameterized by $\vec{r}(u,v)$ for $a \le u \le b$, $c \le v \le d$.

$$\vec{t}_u = \frac{\partial \vec{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

for $a \le u \le b$, ...

Then $\vec{t}_u = \frac{\partial \vec{r}}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$

$\vec{t}_v = \frac{\partial \vec{r}}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$

Then the surface integral of f over S is

$$\iint_S f(x,y,z)\, dS = \iint_R f(\vec{r}(u,v))\, |\vec{t}_u \times \vec{t}_v|\, dA$$

$$= \int_a^b \int_c^d f(\vec{r}(u,v))\, |\vec{t}_u \times \vec{t}_v|\, dv\, du$$

Find the integral of $f(x,y,z) = z$ over the surface of the cylinder $x^2+y^2 = 9$ from $z=0$ to $z=1$

$\vec{r}(u,v) = \langle 3\cos(u), 3\sin(u), v \rangle$ $\quad 0 \le u \le 2\pi$, $\quad 0 \le v \le 1$

$$|\vec{t}_u \times \vec{t}_v| = 3$$

$$\iint_S z\, dS = \iint_R f(\vec{r}(u,v))\, |\vec{t}_u \times \vec{t}_v|\, dA$$

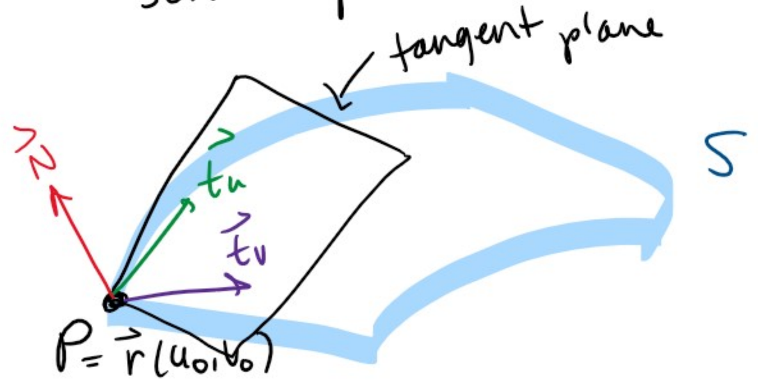
$$= \int_0^{2\pi}\int_0^1 v \cdot 3\, dv\, du = 3\int_0^{2\pi} \left[\frac{v^2}{2}\right]_0^1 du$$

$$= \frac{3}{2}\int_0^{2\pi} du = \frac{3}{2}\cdot 2\pi = \boxed{3\pi}$$

Given a parameterization

$\vec{r}(u,v)$ $\quad a \le u \le b, \quad c \le v \le d$

What is the normal vector at some point (u_0, v_0)?



Normal vector = Normal vector of Tangent Plane

$$\vec{N} = \vec{t}_u \times \vec{t}_v$$