Lesson 32 - 11/13/2023 - 4:30pm

17.6: Surface Integrals (Part 2)

Last Class:

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{t_u}\times\vec{t_v}|\,dA$$

where $\vec{r}(u,v)$ parameterizes S over domain D

and $\vec{t_u} = \frac{\partial \vec{r}}{\partial u}$ and $\vec{t_v} = \frac{\partial \vec{r}}{\partial v}$ ← tangent vectors

Line Integrals: $\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$

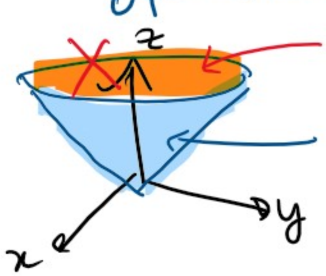
$\vec{r}(t) \longrightarrow \vec{r}\,'$ - tangent vector

Surface Area: Given surface S

$$\text{surface area} = \iint_S 1\cdot dS = \iint_D |\vec{t_u}\times\vec{t_v}|\,dA$$

$\vec{r}(u,v)$ parameterizes S

Example: Find the lateral surface area of the cone $z=\sqrt{x^2+y^2}$ from $z=0$ to $z=3$



don't count the base

lateral surface area

parameterize: cylindrical

$z = r = \sqrt{x^2+y^2}$ $\quad 0 \le z \le 3$

$0 \le \theta \le 2\pi$

let $u=\theta$ $\quad v=z=r$

let $u=\theta$ $v=z=r$

$$\vec{r}(u,v) = \langle r\cos\theta, r\sin\theta, z\rangle = \langle v\cos(u), v\sin(u), v\rangle$$

$0 \le u \le 2\pi$
$0 \le v \le 3$

Lateral Surface area:

$$\iint_D |\vec{t}_u \times \vec{t}_v|\, dA = \int_0^{2\pi}\int_0^3 \sqrt{2}\, v\, dv\, du$$

$$= \sqrt{2}\int_0^{2\pi} \left[\frac{v^2}{2}\right]_0^3 du = \sqrt{2}\cdot\frac{9}{2}\int_0^{2\pi} du$$

$$= \sqrt{2}\,\frac{9}{2}\cdot 2\pi = \boxed{9\pi\sqrt{2}}$$

Mass: Given surface S and a mass-density function $f(x,y,z)$ (in units $g/(cm)^2$)

$$\text{mass} = \iint_S f(x,y,z)\, dS$$

☆ Strategy: - If surface is a plane, leave $\vec{r}(u,v)$ in Cartesian coordinates

- sometimes there are multiple coord systems that work

Ex: (F2007 FE #12)

Find the surface area of the portion of the paraboloid $z = 16 - x^2 - y^2$ that lies above

Find ... paraboloid $z = 16 - x^2 - y^2$ that ... the disk $x^2 + y^2 \le 2$

Method 1: cylindrical coord

$z = 16 - r^2$ $\qquad 0 \le r \le \sqrt{2} \qquad 0 \le \theta \le 2\pi$

let $u = r \qquad v = \theta$

$$\vec{r}(u,v) = \langle u\cos(v),\ u\sin(v),\ 16 - u^2 \rangle$$

$$\vec{t}_u = \langle \cos(v),\ \sin(v),\ -2u \rangle$$

$$\vec{t}_v = \langle -u\sin(v),\ u\cos(v),\ 0 \rangle$$

$$|\vec{t}_u \times \vec{t}_v| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos(v) & \sin(v) & -2u \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix} =$$

$$= |\langle 2u^2\cos(v),\ 2u^2\sin(v),\ \overbrace{u\cos^2(v) + u\sin^2(v)}^{u} \rangle|$$

$$= \sqrt{\underbrace{4u^4\cos^2(v) + 4u^4\sin^2(v)}_{4u^4} + u^2}$$

$$= \sqrt{4u^4 + u^2} = u\sqrt{4u^2+1}$$

$$\iint_S dS = \iint_D u\sqrt{4u^2+1}\, dA = \int_0^{2\pi}\int_0^{\sqrt{2}} u\sqrt{4u^2+1}\, du\, dv$$

Method 2: Cartesian $\qquad$ D - circle of radius $\sqrt{2}$

$$\iint_S 1 \cdot dS = \iint_D \sqrt{1 + 4u^2 + 4v^2}\, dA$$

don't forget

$$\int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{\ \ldots 4r^2}\ r\,dr\,d\theta$$

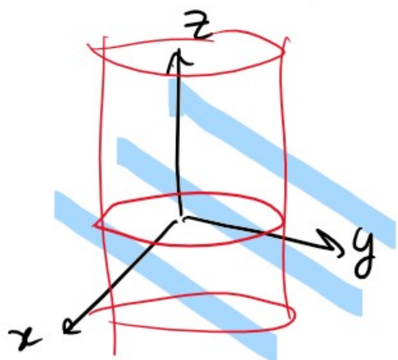
S

D

convert to polar

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{1+4r^2}\; r\,dr\,d\theta$$

Ex: Let S be the portion of the surface $z = 6-y$ that lies inside the cylinder $x^2+y^2=9$



surface is a plane → Cartesian

$$\vec{r}(u,v) = \langle u, v, 6-v \rangle$$

$$\iint_S dS = \iint_D |\vec{t}_u \times \vec{t}_v|\, dA$$

convert to polar
