17.6: Surface Integrals (Part 3)

Scalar Line Integrals:

$$\int_C \underbrace{f(x,y,z)}_{\text{scalar field}}\, ds = \int_a^b f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$$

Scalar Surface Integrals:

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{t}_u \times \vec{t}_v|\, dA$$

Vector Line Integrals

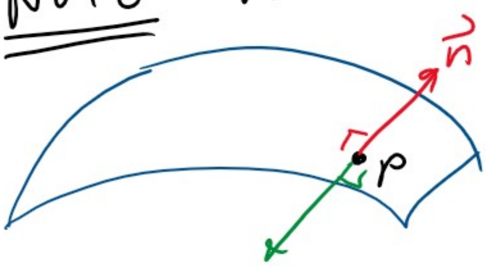
$$\int_C \underbrace{\vec{F}}_{\text{vector field}} \cdot \vec{T}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

Vector Surface Integrals

$$\iint_S \vec{F} \cdot \underbrace{d\vec{S}}_{\text{oriented surface}} = \iint_S \vec{F} \cdot \underbrace{\vec{n}\, dS}_{d\vec{S}}$$

$\vec{n}$ normal vector to the surface

NOTE: A surface has two normal vectors



by convention, we choose $\vec{n}$ to point upwards (z comp. is > 0) or outwards

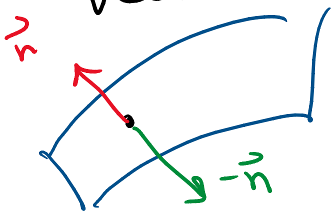
(unless stated otherwise)

Just like vector line integrals, the orientation of curve/surface matters

$$\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r}$$

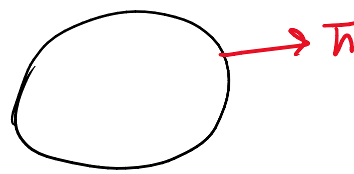
Vector Surface Integrals

Vector Surface Integrals

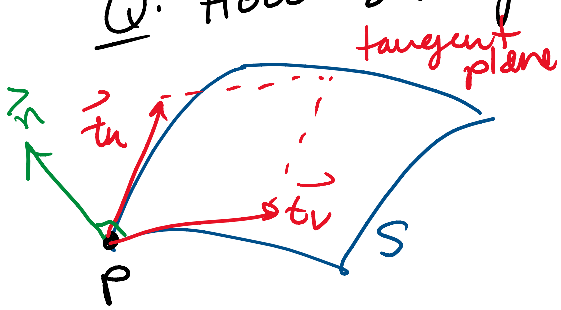


$$\iint_S \vec{F}\cdot\vec{n}\,dS = -\iint_S \vec{F}\cdot(-\vec{n})\,dS$$

If a closed surface is oriented so that $\vec{n}$ points outwards, we say it is <u>positively oriented</u>



<u>Q</u>: How do you find $\vec{n}$?



$\vec{n} \perp S$ at point P

$\vec{n} \perp$ tangent plane

$\vec{n} \perp \vec{t_u}, \vec{t_v}$

The <u>unit normal vector</u>

$$\vec{n} = \frac{\vec{t_u}\times\vec{t_v}}{|\vec{t_u}\times\vec{t_v}|} \quad \text{OR} \quad \frac{\vec{t_v}\times\vec{t_u}}{|\vec{t_v}\times\vec{t_u}|}$$

whichever matches the orientation

Surface Integral

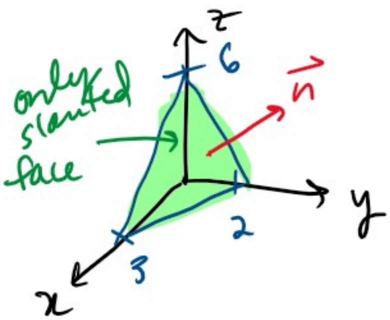
$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D \vec{F}(\vec{r}(u,v))\cdot \underbrace{\frac{(\vec{t_u}\times\vec{t_v})}{\cancel{|\vec{t_u}\times\vec{t_v}|}}}_{\vec{n}} \underbrace{\cancel{|\vec{t_u}\times\vec{t_v}|}\,dA}_{dS}$$

$$= \iint_D \vec{F}(\vec{r}(u,v))\cdot \underbrace{(\vec{t_u}\times\vec{t_v})}_{\text{may need to multiply by } -1 \text{ to get the correct orientation}}\,dA$$

D

may need to ... orientation
to get the correct orientation

<u>Example:</u> Calculate $\iint_S \vec{F}\cdot\vec{n}\, dS$ where $\vec{F} = \langle x, y, z\rangle$ and S is the plane $2x + 3y + z = 6$ in the first octant.



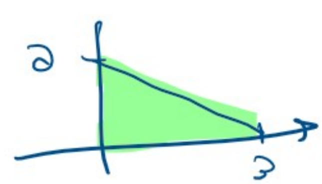
$$\vec{n} = \langle 2, 3, 1\rangle$$

$$\vec{n}\cdot\langle x-x_0,\ y-y_0,\ z-z_0\rangle = 0$$

<u>check:</u>

$$\vec{r}(u,v) = \langle u, v, 6-2u-3v\rangle$$
$$\vec{t}_u = \langle 1, 0, -2\rangle$$
$$\vec{t}_v = \langle 0, 1, -3\rangle$$
$$\vec{t}_u\times\vec{t}_v = \langle 2, 3, 1\rangle \quad \checkmark$$

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_D \vec{F}\cdot(\vec{t}_u\times\vec{t}_v)\, dA$$
$$= \iint_D \langle u, v, 6-2u-3v\rangle\cdot\langle 2,3,1\rangle\, dA$$
$$= \iint_D (\cancel{2u} + \cancel{3v} + 6 - \cancel{2u} - \cancel{3v})\, dA$$
$$= 6\ \text{area}(D)$$
$$= 3\cdot 6 = 18$$



We call $\iint_S \vec{F}\cdot\vec{n}\, dS$ the Flux

If S can be written $z = s(x,y)$

$$\vec{F} = \langle f, g, h \rangle$$

$$\vec{r}(u,v) = \langle u, v, s(u,v) \rangle$$

$$t_u = \langle 1, 0, \tfrac{\partial s}{\partial u} \rangle$$

$$t_v = \langle 0, 1, \tfrac{\partial s}{\partial v} \rangle$$

$$\vec{n} = t_u \times t_v = \langle -\tfrac{\partial s}{\partial u}, -\tfrac{\partial s}{\partial v}, 1 \rangle$$

so
$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_D \langle f, g, h \rangle \cdot \langle -\tfrac{\partial s}{\partial u}, -\tfrac{\partial s}{\partial v}, 1 \rangle\, dA$$

$$= \boxed{\iint_D \left(-f\frac{\partial s}{\partial u} - g\frac{\partial s}{\partial v} + h \right) dA}$$

$z = s(x,y)$