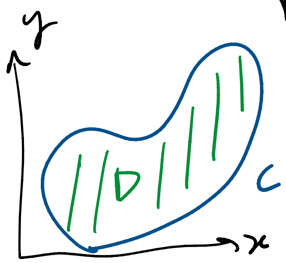
17.7: Stokes' Theorem (Part 1)

Recall: Green's Theorem

$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ in 2D vector field

Let C be a closed curve in xy-plane

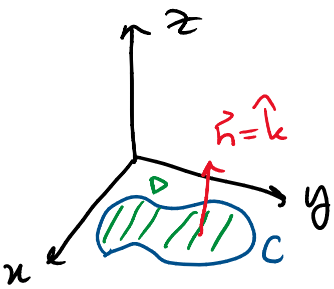
D is the enclosed region



$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

← double integral in xy-plane

Reframe Green's Thm in 3D:



$\vec{F} = \langle P, Q, 0 \rangle$

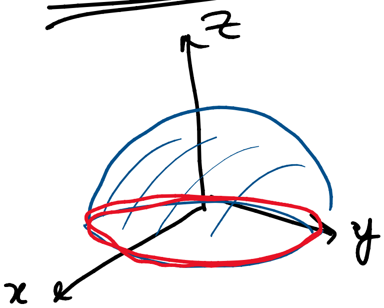
C - curve in xy plane

D - flat region in xy-plane

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\vec{\nabla} \times \vec{F}) \cdot \hat{k} \, dA$$

Q: What if D was a curved surface?

Stokes' Theorem:

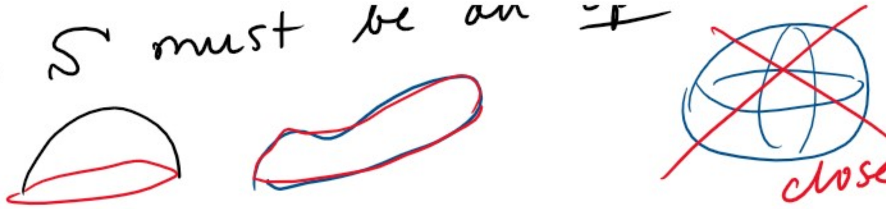


S - surface of a hemisphere

C - curve that outlines opening of S

NOTE: S must be an open surface (it has a hole)

NOTE: S must be an



closed surface

ORIENTATION: Surface S and curve C must be oriented according to the Right Hand Rule

Stokes' Thm: $\vec{F} = \langle P, Q, R \rangle$ 3D vector field

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, dS$$

↑ surface integral

Let $\vec{F} = \langle -y, x, z \rangle$

S - paraboloid $z = x^2 + y^2$ for $z \le 4$

Verify Stokes' Thm:

$$① = \oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, dS = ②$$

Line Integral ① $\vec{r} = \langle 2\cos(t), -2\sin(t), 4 \rangle$

$\vec{r}\,' =$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle 2\sin(t), 2\cos(t), 4 \rangle \cdot \langle -2\sin(t), -2\cos(t), 0 \rangle\, dt$$

$$= \int_0^{2\pi} (-4\sin^2(t) - 4\cos^2(t))\, dt = -4\cdot 2\pi$$

$$= -8\pi$$

② Surface Integral

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & z \end{vmatrix}$$

$$= \langle 0, 0, 1+1\rangle$$
$$= \langle 0, 0, 2\rangle$$

$$\vec{r}(u,v) = \langle u, v, u^2+v^2\rangle$$
$$\vec{t}_u = \langle 1, 0, 2u\rangle$$
$$\vec{t}_v = \langle 0, 1, 2v\rangle$$
$$\vec{t}_u\times\vec{t}_v = \langle -2u, -2v, 1\rangle$$
$$\vec{n} = \langle 2u, 2v, -1\rangle$$

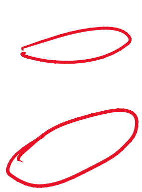
(b/c problem stated downward normal)

$$= \iint_D \langle 0,0,2\rangle\cdot\langle +2u, +2v, -1\rangle\,dA$$

$$= \iint_D -2\,dA = -2\,\text{area}(D)$$
$$= -2\pi(2)^2$$
$$= \boxed{-8\pi}\ ✓$$

Stoke's Thm:

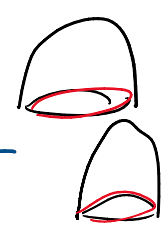
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS$$



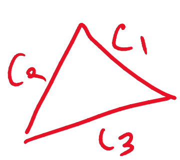
circle
ellipse

might be easier

$\leftarrow$

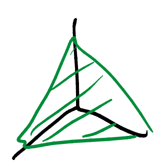


hemisphere
ellipsoid
elliptic paraboloid



triangle

$\longrightarrow$



plane in 1st octant

might be easier

$\vec{F}$ and $\vec{\nabla}\times\vec{F}$, one

☆ Also depends on $\vec{F}$ and $\vec{\nabla} \times \vec{F}$, one might be simpler.

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot (-\vec{n})\, dS = -\iint (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS$$

$$\oint_{-C} \vec{F} \cdot d\vec{r} = -\oint_C \vec{F} \cdot d\vec{r}$$