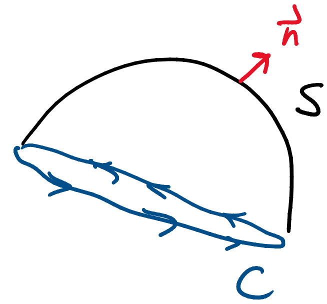
Lesson 35 - 11/27/2023 - 4:30pm

17.7: Stokes' Theorem (Part 2)

Stokes' Thm:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, dS$$

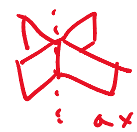


" the fundamental thm of curls "

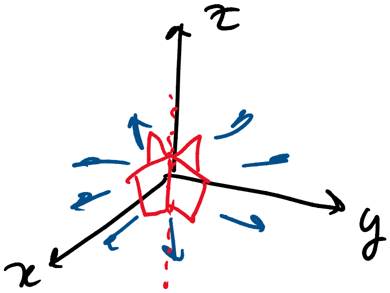
Curl:

Intuition $\vec{F}$ - flow of fluid

Paddle wheel



$\vec{F} = \langle x, y, 0\rangle$



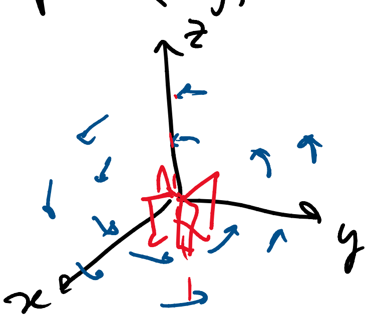
put at the origin
axis of rotation in z-axis

Does not rotate here

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & 0 \end{vmatrix} = \langle 0, 0, 0\rangle$$

no rotation anywhere.

$\vec{F} = \langle -y, x, 0\rangle$



paddle wheel at origin
axis parallel to z-axis

yes, rotates counter clockwise

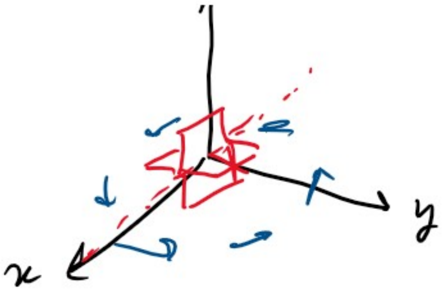
$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = \langle 0, 0, 2\rangle$$

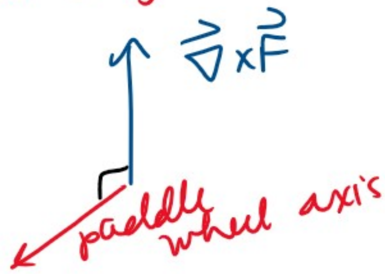
constant rotation around z-axis

paddle at origin axis in x-axis

$\uparrow \vec{\nabla}\times\vec{F}$

rotation



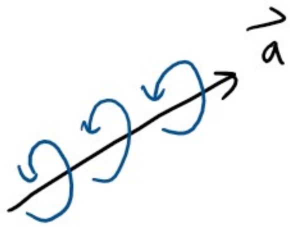


no rotation

<u>Example</u>: General rotatation vector field: $\vec{F} = \vec{a}\times\vec{r}$

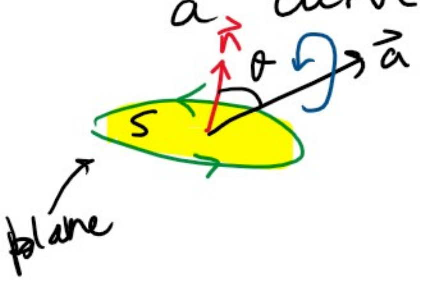
where $\vec{a} = \langle a_1, a_2, a_3 \rangle$ axis of rotation

$\vec{r} = \langle x, y, z \rangle$



$$\vec{\nabla}\times\vec{F} = \vec{\nabla}\times(\vec{a}\times\vec{r}) = 2\vec{a}$$

We can find the <u>average</u> cir<u>culation</u> over a curve C



avg. circulation $\dfrac{1}{\text{area}(S)}\oint_C \vec{F}\cdot d\vec{r}$

Stokes' Thm $= \dfrac{1}{\text{area}(S)}\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS$ ← $\vec{n}$ is constant

$$= \frac{1}{\text{area}(S)}\iint_S 2\vec{a}\cdot\vec{n}\,dS$$

$$= \frac{1}{\text{area}(S)}\,2\vec{a}\cdot\vec{n}\iint_S dS$$

$$= \frac{1}{\cancel{\text{area}(S)}}\,2\vec{a}\cdot\vec{n}\;\cancel{\text{area}(S)}$$

$$= 2|\vec{a}|\,\cancelto{1}{|\vec{n}|}\cos\theta = 2|\vec{a}|\cos\theta$$

max when $\theta = 0$
$\vec{a} \parallel \vec{n}$

zero when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$
$\vec{a} \perp \vec{n}$

min when $\theta = \pi$
$\vec{a} \parallel \vec{n}$
CW

max when $\theta = 0$
$\vec{a} \parallel \vec{n}$
CCW

$\vec{a} \perp \vec{n}$
no circulation

$\vec{a} \parallel \vec{n}$
CW

<u>Thm</u>: If $\vec{\nabla} \times \vec{F} = \vec{0}$ over a region D then $\oint_C \vec{F} \cdot d\vec{r} = 0$
and $\vec{F}$ is conservative

If a vector field $\vec{F}$ is conservative, then all of the following are true:

1) $\vec{F} = \nabla \phi$ $\qquad \phi$ - potential
2) $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ (path independent)
3) $\oint_C \vec{F} \cdot d\vec{r} = 0$
4) $\vec{\nabla} \times \vec{F} = 0$ (irrotational)

<u>Stokes' Thm</u>:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n} \, dS$$

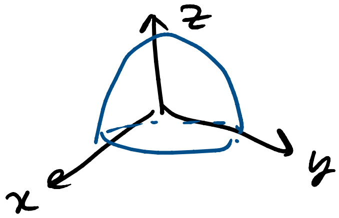
Given curve C — many surfaces S you can use

If C is the circle $x^2 + y^2 = 1$ with $z = 0$
then S could be:

elliptic parabola $\qquad$ sphere $x^2+y^2+z^2 = 1$ $\qquad$ cone $(z-1)^2 = x^2+y^2$

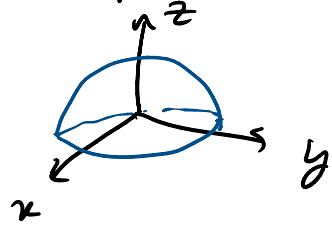
elliptic parabola

$z = 1 - x^2 - y^2$

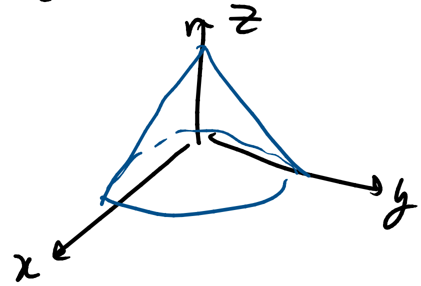


sphere

$x^2 + y^2 + z^2 = 1$



$(z-1)^2 = x^2 + y^2$



Disk

$z=0$