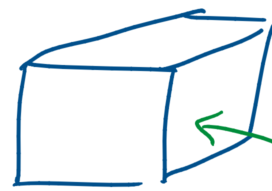
Lesson 36 - 11/29/2023 - 4:30pm

17.8: Divergence Theorem (Part 1)

Divergence:
$$\vec{\nabla}\cdot\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$
$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \leftarrow \text{scalar}$$

Divergence Theorem

$$\iiint_E \underbrace{\vec{\nabla}\cdot\vec{F}}_{\text{scalar}}\, dV = \iint_S \vec{F}\cdot\vec{n}\, dS \leftarrow \text{Flux of } \vec{F} \text{ across } S$$



E volume of cube

S surface of cube.

Intuition:

$\vec{\nabla}\cdot\vec{F}$ is a type of derivative

$\int$ undoes a derivative

$$\iiint_E \underbrace{(\vec{\nabla}\cdot\vec{F})}_{\text{scalar}}\, dV = \iint_S \underbrace{\vec{F}\cdot\vec{n}}_{\text{scalar}}\, dS$$

NOTE: In 2D, $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ the flux form of Green's Thm

Green's Thm (2D): $$\oint_C \vec{F}\cdot\vec{n}\, ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA$$

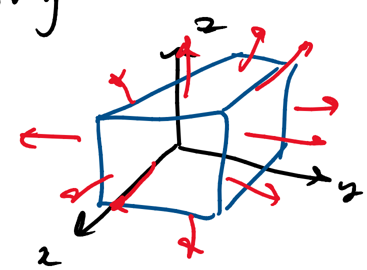
Div Thm (3D): $$\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_E \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV$$

NOTE: Div Thm usually only used for <u>closed</u> surfaces

Q: Why the Div Thm is true?

<u>Surface Flux:</u> $\iint_S \vec{F}\cdot\vec{n}\,dS$

how much of $\vec{F}$ is flowing across S



<u>Ex:</u> $\vec{F} = \langle x, y, z\rangle$

S = box centered at origin

flux should be positive flowing out

<u>Divergence:</u> $\vec{\nabla}\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

how much $\vec{F}$ is expanding

<u>Ex:</u> $\vec{\nabla}\cdot\langle x, y, z\rangle = 1 + 1 + 1 = 3$

$$\iiint_E \vec{\nabla}\cdot\vec{F}\,dV = \iiint_E 3\,dV = 3\,\text{volume}(E)$$

3 b/c expanding in all 3 directions

Consider: $\vec{F} = \frac{\vec{r}}{|\vec{r}|^3} = \frac{\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$

in Gauss' law for the electric field due to a point charge Q at origin in Gauss' law for the electric field due to a point charge Q at origin

$$\vec{E}(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{|\vec{r}|^3}$$

$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m is the permittivity of free space

<u>NOTE</u>: $\vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$ is undefined at $(0,0,0)$

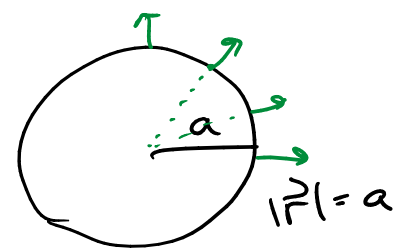
so $\vec{\nabla}\cdot\vec{F}$ is also undefined at $(0,0,0)$

<u>Q</u>: What is $\iint_S \frac{\vec{r}}{|\vec{r}|^3}\cdot\vec{n}\,dS$?

where S is the sphere of radius a.

$\vec{\nabla}\cdot\frac{\vec{r}}{|\vec{r}|^3}$ is undefined in E

so we can't use Div Thm



$\vec{n} \parallel \vec{r}$

unit vector

$\vec{n} = \frac{\vec{r}}{|\vec{r}|}$

$$\iint_S \frac{\vec{r}}{|\vec{r}|^3}\cdot\vec{n}\,dS$$

$$= \iint_S \frac{\vec{r}}{|\vec{r}|^3}\cdot\frac{\vec{r}}{|\vec{r}|}\,dS = \iint_S \frac{|\vec{r}|^{\cancel{2}}}{|\vec{r}|^{\cancel{4}\,2}}\,dS = \frac{1}{a^2}\iint_S dS$$

$$= \frac{1}{a^2}\,\text{surface area (sphere)} = \frac{1}{\cancel{a^2}}\cdot 4\pi\cancel{a^2}$$

$$= 4\pi$$