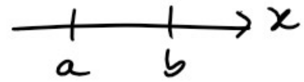
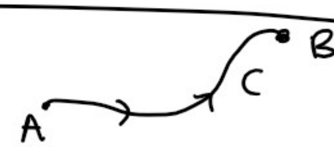
17.8: Divergence Theorem (Part 2)

Fundamental Theorems

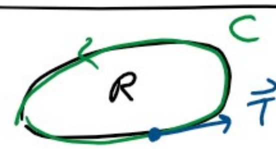
1D Calculus $\int_a^b f'(x)\,dx = f(b) - f(a)$



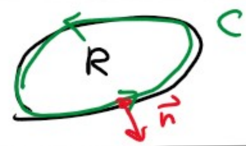
1D curve in 3D space — Line Integrals $\int_C \vec{\nabla} f \cdot d\vec{r} = f(B) - f(A)$



2D Green's Thm (circulation) $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_C \langle P, Q\rangle \cdot \vec{T}\, ds$

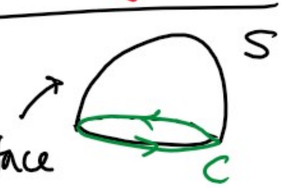


2D Green's Thm (flux) $\iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \int_C \langle P, Q\rangle \cdot \vec{n}\, ds$



3D Stokes' Thm $\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS = \oint_C \vec{F} \cdot d\vec{r}$

open surface



3D Divergence Thm $\iiint_E \vec{\nabla} \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \vec{n}\, dS$

closed surface

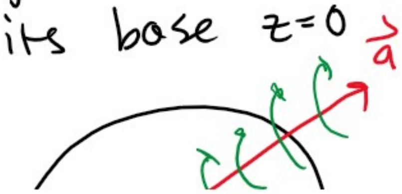
☆ Example: Rotational Field

Let $\vec{F} = \vec{a} \times \vec{r}$ where $\vec{a} = \langle 1, 2, 3\rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ x & y & z \end{vmatrix} = \langle 2z - 3y,\ 3x - z,\ y - 2x\rangle$$

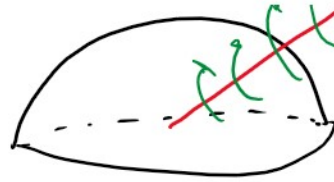
Let S be the hemisphere $x^2 + y^2 + z^2 = a^2$ for $z \geq 0$, together with its base $z = 0$

outward flux



for $z = 0$,

Find the net outward flux



$$\text{flux} = \iint_S \vec{F}\cdot\vec{n}\,dS \overset{\text{Div Thm}}{=} \iiint_E \vec{\nabla}\cdot\vec{F}\,dV$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(2z-3y) + \frac{\partial}{\partial y}(3x-z) + \frac{\partial}{\partial z}(y-2x) = 0$$

rotational field $\rightarrow$ source free $\vec{\nabla}\cdot\vec{F} = 0$

$$= \iiint_E 0\,dV = 0$$

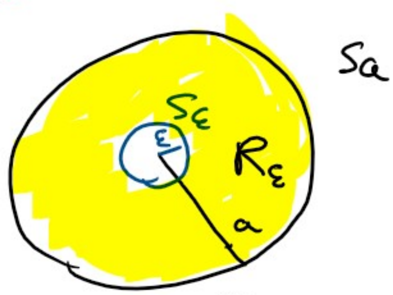
zero flux

Q: What is the flux $\vec{F} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x, y, z\rangle}{\sqrt{x^2+y^2+z^2}}$ over the surface of the sphere of radius a?

Last class $\frac{\vec{r}}{|\vec{r}|^3}$

$\vec{F} = \frac{\vec{r}}{|\vec{r}|}$ is undefined at $(0,0,0)$

Consider



$$\text{Flux} = \iint_{S_a} \frac{\vec{r}}{|\vec{r}|}\cdot\vec{n}\,dS = \lim_{\varepsilon\to 0} \iiint_{R_\varepsilon} \left(\vec{\nabla}\cdot\frac{\vec{r}}{|\vec{r}|}\right) dV$$

$$\vec{\nabla}\cdot\left(\frac{\vec{r}}{|\vec{r}|}\right) = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$\nabla \cdot \left(\frac{\vec{r}}{|\vec{r}|}\right) = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}(\ldots) + \ldots$

$\rho = \sqrt{x^2+y^2+z^2}$

$$= \left(\frac{1}{\rho} + \frac{x\left(-\frac{1}{2}\right)\cdot 2x}{(x^2+y^2+z^2)^{3/2}}\right) + \ldots + \ldots$$

$$= \left(\frac{1}{\rho} - \frac{x^2}{\rho^3}\right) + \left(\frac{1}{\rho} - \frac{y^2}{\rho^3}\right) + \left(\frac{1}{\rho} - \frac{z^2}{\rho^3}\right)$$

$$= \frac{3}{\rho} - \frac{(x^2+y^2+z^2)}{\rho^3} = \frac{3}{\rho} - \frac{\rho^2}{\rho^3}$$

$$= \frac{3}{\rho} - \frac{1}{\rho} = \boxed{\frac{2}{\rho} = \nabla \cdot \frac{\vec{r}}{|\vec{r}|}}$$ as long as $\rho \neq 0$

$$= \lim_{\varepsilon \to 0} \iiint \frac{2}{\rho}\, dV$$

$$= \lim_{\varepsilon \to 0} \int_0^{2\pi} \int_0^{\pi} \int_{\varepsilon}^{a} \frac{2}{\rho}\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \ldots = \lim_{\varepsilon \to 0} 4\pi(a^2 - \varepsilon^2)$$

$$= \boxed{4\pi a^2 = \iint_{S_a} \frac{\vec{r}}{|\vec{r}|} \cdot \vec{n}\, dS}$$

Ex: let R be $x^2+y^2+z^2 \leq 1$ → S is $x^2+y^2+z^2=1$

Use the div. Thm to find

$$\iiint_R z^2\, dV = \iint_S \vec{F} \cdot \vec{n}\, dS$$

$x^2+y^2+y^2=1$

$\vec{n} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x, y, z\rangle}{\rho}$

Need $\vec{\nabla} \cdot \vec{F} = z^2$

$\vec{F} = \left\langle 0, 0, \frac{z^3}{3} \right\rangle$

$\frac{1}{3}\rho^3 \cos^3\phi$

$F = \langle 0, 0, \frac{z^3}{3} \rangle \leftarrow \frac{1}{3}\rho^3 \cos^3\phi$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{z^3}{3} \cdot \frac{z}{\rho} \rho^2 \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi}\int_0^{\pi} \cancel{\rho^6} \frac{\cos^4(\phi) \sin\phi}{\cancel{\rho}} \, d\phi \, d\theta$$

$\rho = 1$ on S