☆Dot Products

Q: How can we multiply 2 vectors?

A: Two ways

(Today) dot product: $\vec{u} \cdot \vec{v} \rightarrow$ scalar

(Next Class) cross product: $\vec{u} \times \vec{v} \longrightarrow$ vector

Ex: $\vec{u} = \langle 5, 0, 3 \rangle$
$\vec{v} = \langle 0, 7, -1 \rangle$

dot product: $\vec{u} \cdot \vec{v} = \langle 5, 0, 3 \rangle \cdot \langle 0, 7, -1 \rangle$
$= 5 \cdot 0 + 0 \cdot 7 + 3 \cdot (-1)$
$= 0 + 0 + -3$

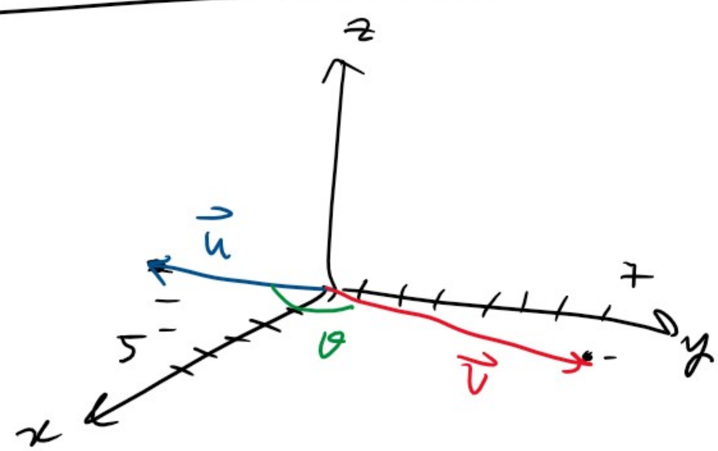
$$\boxed{\vec{u} \cdot \vec{v} = -3}$$

Def: Given $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3} \leftarrow \text{scalar}$$

Geometrically:

$\vec{u} = \langle 5, 0, 3 \rangle$
$\vec{v} = \langle 0, 7, -1 \rangle$



nonzero vectors

$\underline{\text{Thm}}$: If $\vec{u}$ and $\vec{v}$ are nonzero vectors then $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$ where $0 \le \theta \le \pi$

$|\vec{u}| = \sqrt{5^2+0^2+3^2} = \sqrt{25+9} = \sqrt{34}$

$|\vec{v}| = \sqrt{0^2+7^2+(-1)^2} = \sqrt{49+1} = \sqrt{50}$

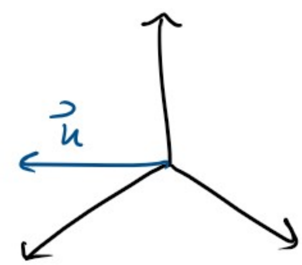
$$-3 = \vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

$$-3 = \sqrt{34}\cdot\sqrt{50}\cos\theta$$

$$\frac{-3}{10\sqrt{17}} = \frac{-3}{\sqrt{34\cdot 50}} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{-3}{10\sqrt{17}}\right) \approx 94.2^\circ$$

$\underline{\text{Ex}}$: $\vec{u} = \langle 5, 0, 3\rangle$ What is $\vec{u}\cdot\vec{u} = ?$



$$5\cdot 5 + 0\cdot 0 + 3\cdot 3 = \vec{u}\cdot\vec{u} = |\vec{u}||\vec{u}|\cos\theta$$

$$\frac{34}{34} = 25 + 9 = \frac{\left(\sqrt{34}\right)^2\cos\theta}{34}$$

$$1 = \cos\theta$$

$$\theta = \cos^{-1}(1) = 0 \quad ✓$$

What if $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\sin\theta$?

If $\vec{v} = \vec{u}$ $\quad |\vec{u}|^2 = \vec{u}\cdot\vec{u} = |\vec{u}||\vec{u}|\sin\theta$ $\longrightarrow$ $\theta = \frac{\pi}{2}$ $\longrightarrow$ $\sin\theta = 1$

NOTE: $\vec{u}\cdot\vec{u} = |\vec{u}|^2$

$\underline{\text{Def}}$: Two vectors $\vec{u}$ and $\vec{v}$ are <u>orthogonal</u> if and only if $\vec{u}\cdot\vec{v} = 0$

only if $\vec{u} \cdot \vec{v} = 0$

$\vec{u}, \vec{v}$ orthogonal $\iff$ $\vec{u} \cdot \vec{v} = 0$

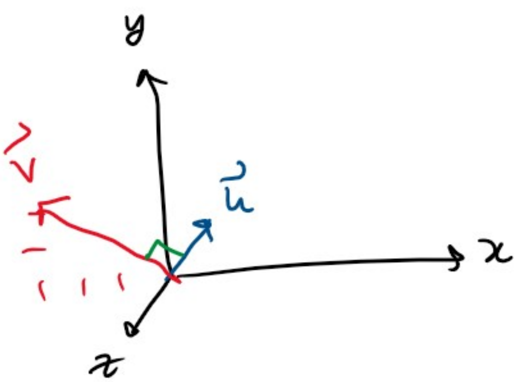
$\vec{u} \perp \vec{v}$

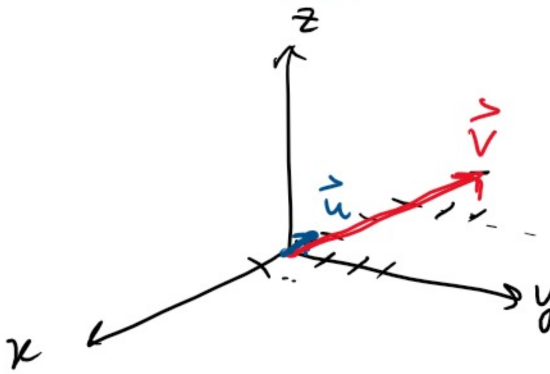
<u>Ex:</u> $\vec{u} = \langle 1, 1, 1 \rangle$

$\vec{v} = \langle -3, 2, 1 \rangle$

$\vec{u} \cdot \vec{v} = 1 \cdot (-3) + 1 \cdot 2 + 1 \cdot 1 = -3 + 2 + 1$

$\vec{u} \cdot \vec{v} = 0 \implies$ $\vec{u}$ and $\vec{v}$ are orthogonal





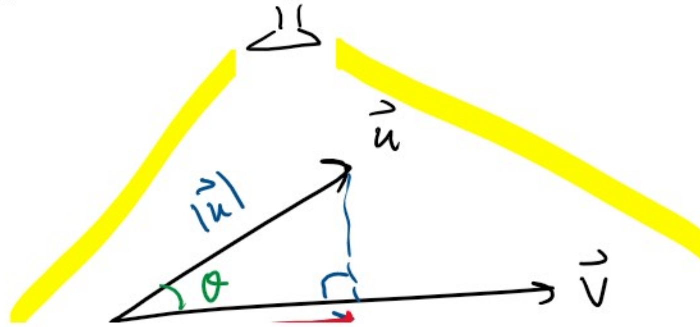
<u>Applications of the Dot Product</u>

<u>Q</u>: Given $\vec{u}$ and $\vec{v}$, how much of $\vec{u}$ points in the direction of $\vec{v}$?

(how closely aligned are they?)

$\Downarrow$

<u>Geometrically:</u>



"shadow" of u

$\text{proj}_{\vec{v}}(\vec{u}) \parallel \vec{v}$

$$\text{proj}_{\vec{v}}(\vec{u}) = \underbrace{\left(|\vec{u}|\cos\theta\right)}_{\text{magnitude of } \text{proj}_{\vec{v}}(\vec{u})} \underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\text{unit vector in the direction}}$$

trig:

$$\cos\theta = \frac{|\text{proj}_{\vec{v}}(\vec{u})|}{|\vec{u}|}$$

$$|\vec{u}|\cos\theta = |\text{proj}_{\vec{v}}(\vec{u})|$$

$$= \text{scal}_{\vec{v}}(\vec{u})$$

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

$$\text{proj}_{\vec{v}}(\vec{u}) = \left(\frac{|\vec{u}||\vec{v}|\cos\theta}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|}$$

$$= \left(\frac{|\vec{u}||\vec{v}|\cos\theta}{|\vec{v}|^2}\right)\vec{v}$$

$$\boxed{|\vec{u}|\cos\theta\,\frac{\vec{v}}{|\vec{v}|} = \text{proj}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}}$$

Ex: $\vec{u} = \langle 7, -3, 2\rangle$

$\vec{v} = \langle 1, 0, 0\rangle$

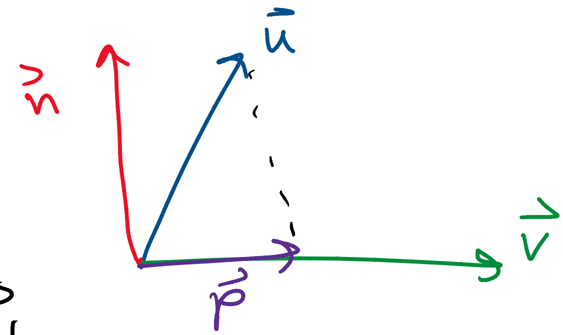
$$\text{proj}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

$$= \frac{7\cdot 1 + (-3)\cdot 0 + 2\cdot 0}{1\cdot 1 + 0\cdot 0 + 0\cdot 0}\langle 1, 0, 0\rangle$$

$$= 7\langle 1, 0, 0\rangle = \boxed{\langle 7, 0, 0\rangle}$$

$$= \frac{7}{1}\langle 1,0,0\rangle = \boxed{\langle 7,0,0\rangle}$$

Ex: $\vec{u} = \langle 5,1,2\rangle$
$\vec{v} = \langle 0,1,1\rangle$



$\vec{u} = \vec{p} + \vec{n}$

Decompose $\vec{u} = \vec{p} + \vec{n}$

where $\vec{p} \parallel \vec{v}$
$\vec{n} \perp \vec{v}$

Find $\vec{p} = \text{proj}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$

$$= \left(\frac{5\cdot 0 + 1\cdot 1 + 2\cdot 1}{0\cdot 0 + 1\cdot 1 + 1\cdot 1}\right)\langle 0,1,1\rangle$$

$$= \frac{3}{2}\langle 0,1,1\rangle = \boxed{\langle 0, \tfrac{3}{2}, \tfrac{3}{2}\rangle = p}$$

$(-\vec{p})$ $\quad \vec{u} = \vec{p} + \vec{n} - \vec{p}$

$\vec{u} - \vec{p} = \boxed{\vec{n} = \vec{u} - \vec{p}}$

$\vec{n} = \langle 5,1,2\rangle - \langle 0, \tfrac{3}{2}, \tfrac{3}{2}\rangle$

$\boxed{\vec{n} = \langle 5, -\tfrac{1}{2}, \tfrac{1}{2}\rangle}$

$$\boxed{\begin{gathered}\vec{u} = \vec{p} + \vec{n} \\ \langle 5,1,2\rangle = \langle 0, \tfrac{3}{2}, \tfrac{3}{2}\rangle + \langle 5, -\tfrac{1}{2}, \tfrac{1}{2}\rangle\end{gathered}}$$

$\vec{n} \perp \vec{v}$ ✓

$\vec{n}\cdot\vec{v} = 0$

$\langle 5, 1, 2 \rangle =$ $\qquad\qquad$ $\vec{n} \perp \vec{v}$ ✓

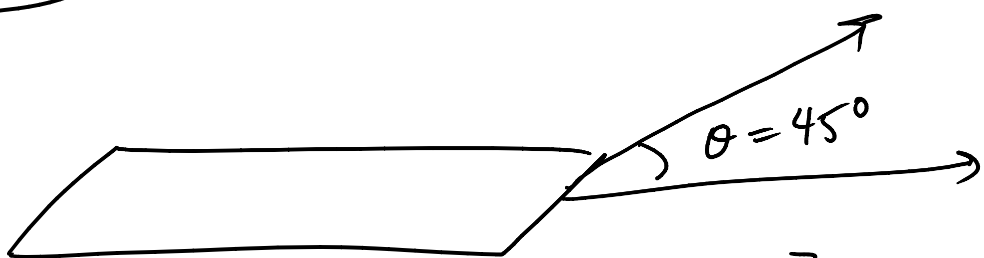
Check $\vec{n} \cdot \vec{v} = 0$

$\langle 5, \frac{-1}{2}, \frac{1}{2} \rangle \cdot \langle 0, 1, 1 \rangle = 5 \cdot 0 + \frac{-1}{2} \cdot 1 + \frac{1}{2} \cdot 1$
$= 0$

<u>Work</u>: Sled

$|\vec{F}| = 5N$

$|\vec{d}| = 10m$



distance 10m

What is the work done?

$$\vec{W} = \vec{F} \cdot \vec{d} = |\vec{F}||\vec{d}| \cos\theta$$
$$= 5 \cdot 10 \cos(45°)$$
$$\boxed{W = 50 \cdot \frac{\sqrt{2}}{2}}$$