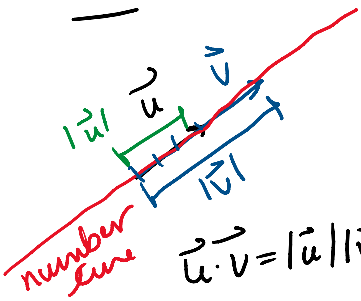
Vector Multiplication:

Two Types: Dot Product: $\vec{u} \cdot \vec{v} = \text{scalar}$

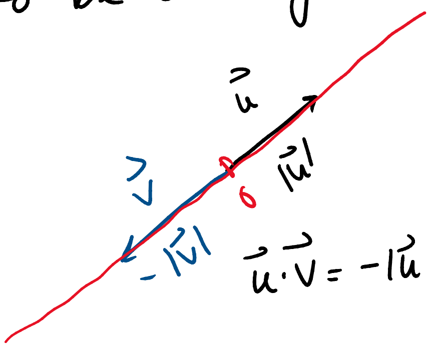
Cross Product $\vec{u} \times \vec{v} = \text{vector}$

Q: Why does the dot product have $\cos\theta$?

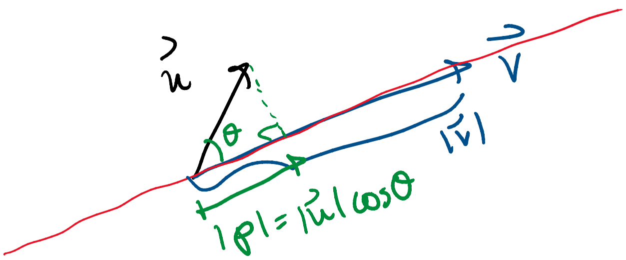
A: Want it to be analogous to scalar multiplication



$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|$

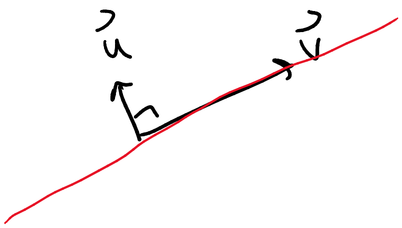


$\vec{u} \cdot \vec{v} = -|\vec{u}||\vec{v}|$



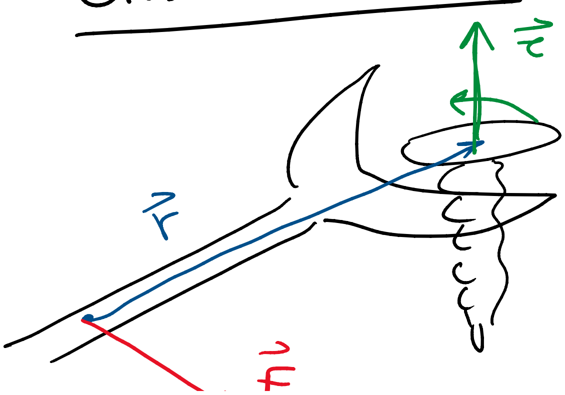
$\vec{u} \cdot \vec{v} = |\vec{v}||p|$

$= |\vec{v}||\vec{u}|\cos\theta$



$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos(90^\circ) = 0$

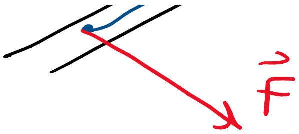
Cross Products: Inspired by torque in physics



lefty loosey
righty tighty

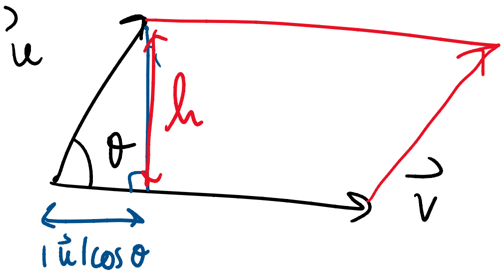
$\vec{r} \times \vec{F} = \vec{\tau}$

both $\vec{r}$ and $\vec{F}$



$\vec{\tau} \perp$ both $\vec{r}$ and $\vec{F}$

So $\vec{w} = \vec{u} \times \vec{v}$ then $\vec{w} \perp \vec{u}$ and $\vec{v}$



parallelogram

$|\vec{u} \times \vec{v}| =$ area of the parallelogram

$=$ base $\times$ height

$= |\vec{v}| \cdot |\vec{u}| \sin\theta$

$\sin\theta = \dfrac{h}{|\vec{u}|}$

$h = |\vec{u}| \sin\theta$

$$\boxed{|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta}$$

$\vec{u} \times \vec{v}$ — a vector
— orthogonal to $\vec{u}$ and $\vec{v}$ ← 2 possible directions
— magnitude is $|\vec{u}||\vec{v}|\sin\theta$

Q: Which direction to choose?
→ Right Hand Rule

Note: $\hat{i} \times \hat{j} = \hat{k}$

Calculating Cross Products:

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Def: the determinant of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Ex: $|B| = \begin{vmatrix} 5 & 7 \\ 1 & -1 \end{vmatrix} = 5\cdot(-1) - 7\cdot 1 = -5-7 = \boxed{-12}$

Cross Product: $\vec{u} = \langle 2, -1, 0 \rangle$

$\vec{v} = \langle 1, 2, 3 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 0 \\ 1 & 2 & 3 \end{vmatrix} \begin{array}{l} \\ \leftarrow \vec{u} \\ \leftarrow \vec{v} \end{array}$$

Expansion by Minors

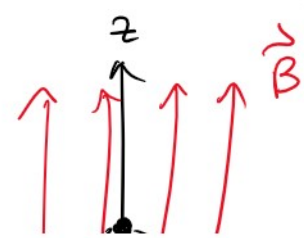
sign alternates +, −, +

$$= \hat{\imath} \begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} + k \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= \hat{\imath}\left((-1)\cdot 3 - \cancelto{0}{0\cdot 2}\right) - \hat{\jmath}\left(2\cdot 3 - \cancelto{0}{0\cdot 1}\right) + \hat{k}\left(2\cdot 2 - (-1)\cdot 1\right)$$

$$= -3\hat{\imath} - 6\hat{\jmath} + \hat{k}(4+1) = \boxed{\langle -3, -6, 5 \rangle}$$

Application: Force on a charge



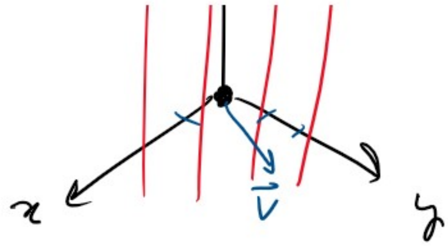
A particle with charge $q = -1$

... $\vec{B} = 5\hat{k}$

A particle with charge $q = -1$

Constant magnetic field $\vec{B} = 5\hat{k}$

particle velocity $\vec{v} = \hat{\imath} + 2\hat{\jmath}$



Force on particle: $\vec{F} = q(\vec{v} \times \vec{B})$

$$\vec{F} = \overset{q}{(-1)} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 0 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= (-1)\left\{ \hat{\imath} \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & 0 \\ 0 & 5 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \right\}$$

$$= (-1)\left\{ \hat{\imath}(10) - \hat{\jmath}(5) + \hat{k}\,0 \right\}$$

$$= \boxed{\langle -10, 5, 0 \rangle = \vec{F}}$$

<u>Ex:</u> Find all the vectors $\vec{u}$ that satisfy

$$\langle 1, 1, 1 \rangle \times \vec{u} = \langle -1, -1, 2 \rangle$$

Let $\vec{u} = \langle x, y, z \rangle$

$$\langle 1, 1, 1 \rangle \times \langle x, y, z \rangle = \langle -1, -1, 2 \rangle$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \end{vmatrix} = \langle -1, -1, 2 \rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \langle -1, -1, 2 \rangle$$

$$\hat{i}\begin{vmatrix} 1 & 1 \\ y & z \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 1 \\ x & z \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ x & y \end{vmatrix} = \langle -1, -1, 2 \rangle$$

$$\hat{i}(z-y) - \hat{j}(z-x) + \hat{k}(y-x) = \langle -1, -1, 2 \rangle$$

$$\langle z-y,\ x-z,\ y-x \rangle = \langle -1, -1, 2 \rangle$$

system of 3 equations + 3 unknowns

$$\begin{cases} ① & z - y = -1 \\ ② & x - z = -1 \\ ③ & y - x = 2 \end{cases}$$

$y - x = 2$

$\boxed{y = 2 + x}$

substitute $y = 2+x$ into ①

$$z - y = -1$$
$$z - (2+x) = -1$$
$$z = -1 + (2+x) = x + 1$$

$\boxed{z = x+1}$

substitute $z = x+1$ into ②

$$x - z = -1$$
$$x - (x+1) = -1$$
$$-1 = -1$$

eqn ② is a duplicate of eqns ① and ③

$x - (x+1) = -1$

$-1 = -1$

eqns' ① and ③

→ NO unique solution

One free variable

$z = x+1$

$y = x+2$

Choose x to be free variable

"independent variable"

$$\boxed{\vec{u} = \langle x, y, z \rangle = \langle x, x+2, x+1 \rangle \quad \text{where } x \text{ is any real numbers}}$$