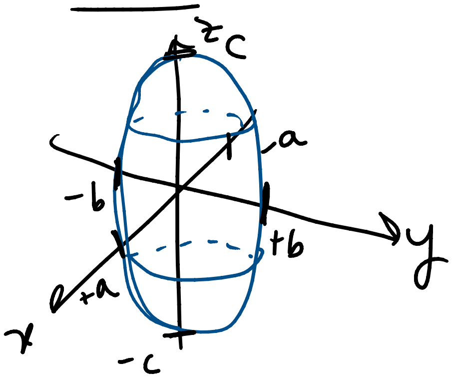
Lesson 5 - 8/30/2023 - 4:30pm

Sec 14.1: Vector-Valued Functions

NOTE: Lengths of Ellipsoid



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

length in x-dim: $2a$

y-dim: $2b$

z-dim: $2c$

Def: A vector-valued function is a function of the form:

$$\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$
$$= \langle f(t), g(t), h(t) \rangle$$

where the component functions f, g, h are real-valued functions of the parameter t

Ex: $\vec{r}(t) = \langle 1,2,3 \rangle + t\langle 4,5,6 \rangle$
$= \langle 1+4t, 2+5t, 3+6t \rangle$ ← describes a line

Ex: $\vec{r}(t) = \langle 4\cos(t), 3\sin(t), \frac{1}{t} \rangle$

Evaluate $\vec{r}\left(\frac{\pi}{2}\right)$ and $\vec{r}(0)$

$\vec{r}\left(\frac{\pi}{2}\right) = \vec{r}(t)\Big|_{t=\frac{\pi}{2}} = \left\langle 4\cos\left(\frac{\pi}{2}\right), 3\sin\left(\frac{\pi}{2}\right), \frac{1}{\frac{\pi}{2}} \right\rangle$

$$\vec{r}\left(\tfrac{\pi}{2}\right) = \vec{r}(t)\Big|_{t=\frac{\pi}{2}} = \left\langle 4\cos\left(\tfrac{\pi}{2}\right), 3\sin\left(\tfrac{\pi}{2}\right), \tfrac{1}{\pi/2} \right\rangle$$
$$= \left\langle 0, 3, \tfrac{2}{\pi} \right\rangle$$

$$\vec{r}(0) = \vec{r}(t)\Big|_{t=0} = \left\langle 4\cos(0), 3\sin(0), \tfrac{1}{0} \right\rangle$$
$$= \langle 4, 0, \Rightarrow\Leftarrow \rangle$$

↑ undefined

$\vec{r}(0)$ is undefined @ $t=0$

Def: The domain of a vector-valued function is the largest set of values for which $f(t), g(t), h(t)$ are all defined

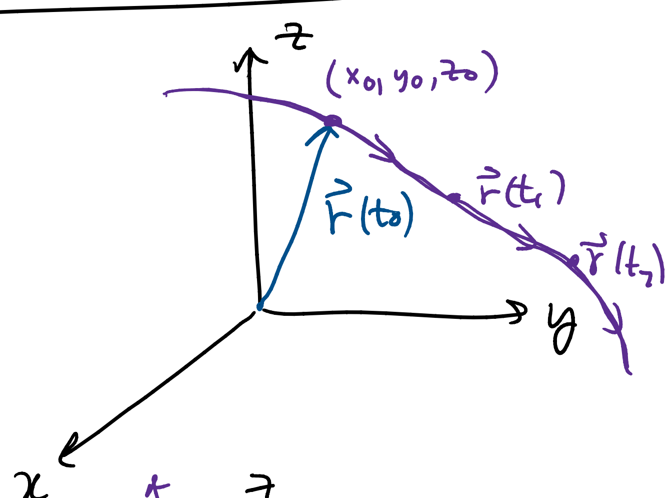
fcn	domain
$4\cos(t)$	$(-\infty, \infty)$
$3\sin(t)$	$(-\infty, \infty)$
$\frac{1}{t}$	$t \neq 0$ $\quad (-\infty, 0), (0, \infty)$

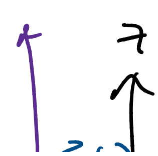
$$\boxed{\text{Domain } \vec{r}(t): \quad (-\infty, 0), (0, \infty)}$$

Graphing

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

t	$\vec{r}(t)$
t_0	$\langle f(t_0), g(t_0), h(t_0) \rangle = \vec{r}(t_0)$
t_1	$\vec{r}(t_1)$
t_2	$\vec{r}(t_2)$



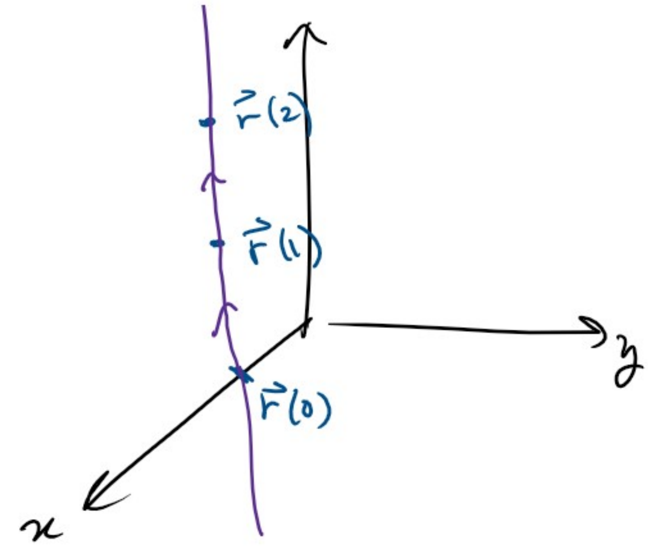


$t_2 \mid \vec{r}(t_2)$

<u>Example:</u>

$\vec{r}(t) = \langle 1,0,0 \rangle + t\langle 0,0,3 \rangle$
$= \langle 1,0,3t \rangle$

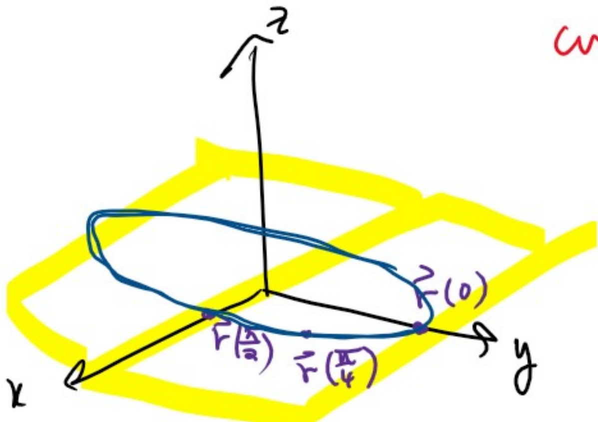
t	$\vec{r}(t)$
0	$\langle 1,0,0 \rangle$
1	$\langle 1,0,3 \rangle$
2	$\langle 1,0,6 \rangle$



<u>Example:</u>

$\vec{r}(t) = \langle \sin(t), 3\cos(t), 0 \rangle$

$z=0$ is always zero
curve lies in xy-plane



t	$\vec{r}(t)$
0	$\langle 0,3,0 \rangle$
$\frac{\pi}{4}$	$\langle \frac{\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, 0 \rangle$
$\frac{\pi}{2}$	$\langle 1,0,0 \rangle$

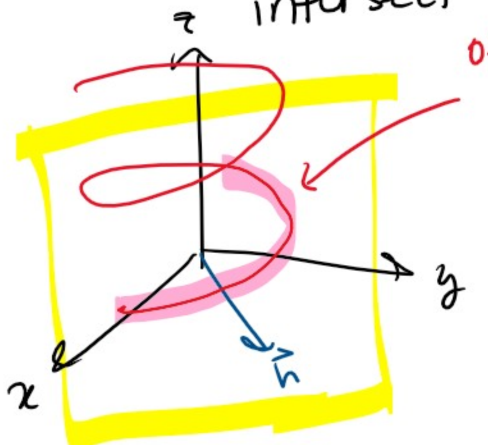
<u>Ellipse</u> $\longrightarrow$ think of t as angle θ

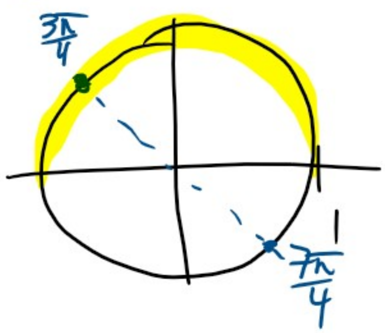
<u>Intersections:</u>

<u>Q</u>: Find the point(s) where the plane $x+y=0$ and curve $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ $(0 \le t \le \pi)$

$\rightarrow$ intersect

intersect





$x + y = 0$

$y = -x$

$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$

$= \langle x, y, z \rangle$

$\sin(t) = -\cos(t)$ $\quad 0 \le t \le \pi$

$\tan(t) = \dfrac{\sin(t)}{\cos(t)} = -1$

$t = \frac{3\pi}{4}$

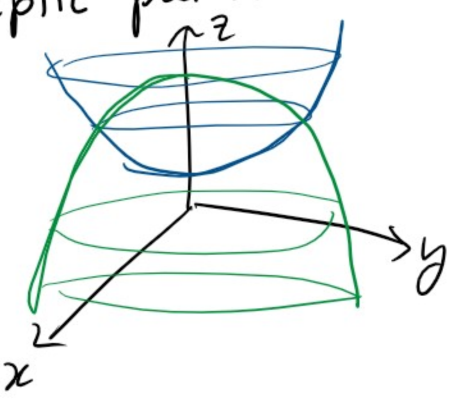
point of intersection

$\vec{r}\left(\frac{3\pi}{4}\right) = \left\langle \cos\left(\frac{3\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right), \frac{3\pi}{4} \right\rangle$

$\boxed{\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)}$

Example: Find the curve that describes the intersection of the elliptic parabolas:

$$\begin{cases} z = 3x^2 + y^2 + 1 \\ z = 5 - x^2 - 3y^2 \end{cases}$$

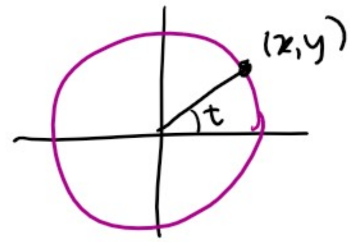


$z = 3x^2 + y^2 + 1$

$- \quad z = 5 - x^2 - 3y^2$

$4x^2 + 4y^2 = 4$

$$0 = 4x^2 + 4y^2 - 4$$
$$4x^2 + 4y^2 = 4$$
$$x^2 + y^2 = 1$$ ← equation of a circle



$$y^2 = 1 - x^2$$

$$z = 5 - x^2 - 3y^2$$
$$= 5 - x^2 - 3(1 - x^2)$$
$$= 5 - x^2 - 3 + 3x^2$$
$$z = 2 + 2x^2$$

$$\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$$

$$\langle x, y, z \rangle = \langle x, \pm\sqrt{1 - x^2}, 2 + 2x^2 \rangle$$

$$\boxed{\vec{r}(t) = \langle \cos(t), \sin(t), 2 + 2\cos^2(t) \rangle}$$

$$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$$

$$\begin{cases} x = 2\cos(t) \\ y = 1 \cdot \sin(t) \end{cases}$$