Lesson 6 - 9/1/2023 - 4:30pm

Sec 14.2: Calculus of Vector-Valued Functions

☆ Derivatives:

Def: The derivative of a vector-valued function $\vec{r}(t)$ is

$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

Example: Let $\vec{r}(t) = \langle 3t+4, 2 \rangle$

Find $\vec{r}\,'(t) = \lim_{\Delta t \to 0} \dfrac{\langle 3(t+\Delta t)+4, 2\rangle - \langle 3t+4, 2\rangle}{\Delta t}$

$$= \lim_{\Delta t \to 0} \frac{\langle \cancel{3t} + 3\Delta t \cancel{+4} \cancel{-3t} \cancel{-4}, \cancel{2} - \cancel{2} \rangle}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\langle 3\Delta t, 0\rangle}{\Delta t} = \lim_{\Delta t \to 0} \langle 3, 0 \rangle$$

$$\boxed{\vec{r}\,'(t) = \langle 3, 0 \rangle}$$

NOTE: $\vec{r}\,'(t)$ looks like $\langle \frac{d}{dt}(3t+4), \frac{d}{dt}(2) \rangle$

Cor: If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ and f, g, h are differentiable at t, then $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Example: $\vec{r}(t) = \langle t\ln(t), 5e^t, \cos(t) - \sin(t) \rangle$

Find $\vec{r}\,'(t) = \langle \frac{d}{dt}(t\ln(t)), \frac{d}{dt}(5e^t), \frac{d}{dt}(\cos t - \sin t) \rangle$

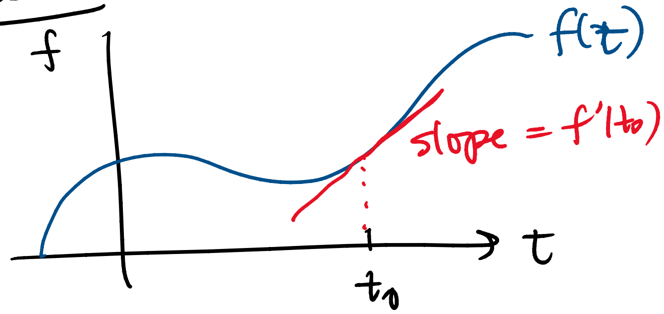
Find $\vec{r}\,'(t) = \left\langle \frac{d}{dt}(t\ln(t)), \frac{d}{dt}(5e^t), \frac{d}{dt}(\cos \ldots) \right\rangle$

$$= \langle 1 \cdot \ln(t) + t\left(\tfrac{1}{t}\right), 5e^t, -\sin(t) - \cos(t) \rangle$$

$$= \boxed{\langle \ln(t) + 1, 5e^t, -\sin(t) - \cos(t) \rangle}$$

Tangent Vectors + Tangent Lines:

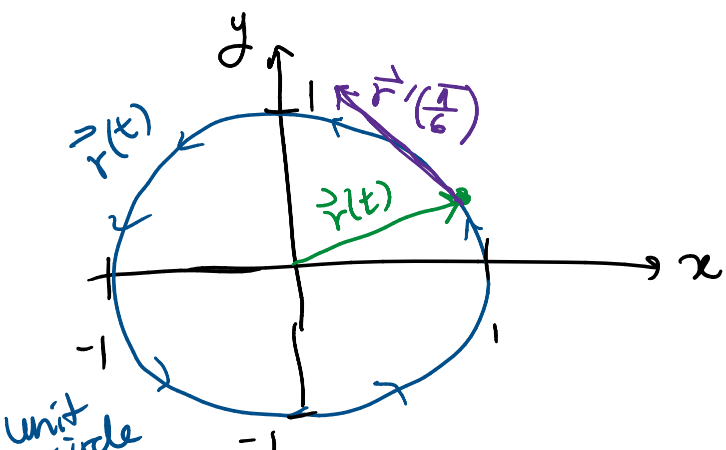
Recall: function of 1 variable



$f'(t_0)$ is the instaneous slope at $t = t_0$

Vector-valued functions:

$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$



$$\vec{r}\left(\tfrac{\pi}{6}\right) = \left\langle \cos\left(\tfrac{\pi}{6}\right), \sin\left(\tfrac{\pi}{6}\right) \right\rangle$$
$$= \left\langle \tfrac{\sqrt{3}}{2}, \tfrac{1}{2} \right\rangle$$

$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$$

$$\vec{r}\,'\left(\tfrac{\pi}{6}\right) = \left\langle -\tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \right\rangle$$

NOTE: $\vec{r}\,'\left(\frac{\pi}{6}\right)$ is tangent to the circle at $t = \frac{\pi}{6}$

We call $\vec{r}\,'\left(\frac{\pi}{6}\right)$ the tangent vector to the curve $\vec{r}(t)$ at $t = \frac{\pi}{6}$

we can
the curve $\vec{r}(t)$ at $t=\frac{\pi}{6}$

Def: Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is differentiable for all $a \le t \le b$. Then the unit tangent vector $\vec{T}(t)$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

makes it a unit vector

☆ Tangent Line:

Find the line tangent to $\vec{r}(t) = \langle e^{t}, e^{2t}, e^{3t} \rangle$ at the point $t=0$.

1. Find tangent vector @ $t=0$

$$\vec{r}\,'(t)\Big|_{t=0} = \langle e^{t}, 2e^{2t}, 3e^{3t} \rangle \Big|_{t=0} = \langle 1, 2, 3 \rangle$$

direction vector

2. Find a point: $\vec{r}(t=0)$

$$\vec{r}(t=0) = \langle e^{0}, e^{2\cdot 0}, e^{3\cdot 0} \rangle \rightarrow (1, 1, 1)$$

point

3. Tangent line:

$$\langle x, y, z \rangle = \langle 1, 1, 1 \rangle + t\langle 1, 2, 3 \rangle$$
$$= \boxed{\langle 1+t, 1+2t, 1+3t \rangle}$$

☆ Integrals:

Let $\vec{r}(t)$ be a vector-valued function

☆ Integrals:

Let $\vec{r}(t)$ be a vector-valued function whose derivative is $\vec{r}'(t)$, then

$$\int_a^b \vec{r}'(t)\,dt = \vec{r}(b) - \vec{r}(a)$$

$$\int \vec{r}'(t)\,dt = \vec{r}(t) + \vec{c}$$

constant vector of integration

$\vec{c} = \langle c_1, c_2, c_3 \rangle$

Example:

$$\int_1^2 \langle 2t, e^t \rangle\,dt = \left[\langle t^2, e^t \rangle\right]\Big|_{t=1}^{t=2}$$

$$= \langle 2^2, e^2 \rangle - \langle 1^2, e^1 \rangle$$

$$= \langle 4-1, e^2 - e \rangle = \boxed{\langle 3, e^2 - e \rangle}$$

☆ Motion in Space:

t – time

$\vec{r}(t)$ – position of an object in space

$\vec{v}(t) = \vec{r}'(t)$ – velocity

$|\vec{r}'(t)|$ – speed (scalar)

$\vec{a}(t) = \vec{r}''(t)$ – acceleration
$= \vec{v}'(t)$

NOTE: we can move between these by taking derivatives or integrals

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Example: If $\vec{a}(t) = \langle \sin(t), 2, t \rangle$ is acceleration

... $\vec{v}(t)$ so that $\vec{v}(t=0) = \langle 1, 2, 3 \rangle$

Example: If $\vec{a}(t) = \langle \sin(t), 2, t \rangle$

Find the velocity $\vec{v}(t)$ so that $\underbrace{\vec{V}(t=0) = \langle 1, 2, 3 \rangle}_{\text{initial condition}}$

$$\int \vec{a}(t) = \vec{r}''(t) = \int \vec{V}'(t)$$

$$\vec{V}(t) = \int \vec{a}(t)\,dt = \int \langle \sin(t), 2, t \rangle\,dt$$

$$= \left\langle \int \sin(t)\,dt,\ \int 2\,dt,\ \int t\,dt \right\rangle$$

$$\star \quad = \left\langle -\cos(t) + c_1,\ 2t + c_2,\ \frac{t^2}{2} + c_3 \right\rangle$$

$$= \left\langle -\cos(t), 2t, \frac{t^2}{2} \right\rangle + \underbrace{\langle c_1, c_2, c_3 \rangle}_{\vec{c}}$$

constant vector of integration.

initial condition

Want $\vec{V}(t=0) = \langle 1, 2, 3 \rangle = \left\langle -\cos(t) + c_1, 2t + c_2, \frac{t^2}{2} + c_3 \right\rangle \Big|_{t=0}$

$$\langle 1, 2, 3 \rangle = \langle -1 + c_1, c_2, c_3 \rangle$$

$1 = -1 + c_1 \qquad 2 = c_2 \qquad 3 = c_3$

$c_1 = 2$

$$\boxed{\vec{V}(t) = \left\langle -\cos(t) + 2,\ 2t + 2,\ \frac{t^2}{2} + 3 \right\rangle}$$

$$\int_0^t \vec{a}(t) = \vec{V}(t) - \vec{V}(0)$$

should give a similar answer.