Lesson 7 - 9/6/2023 - 4:30pm

14.3: Motion in Space

$$\int_0^t \vec{a}(s)ds = \vec{v}(t) - \vec{v}(0)$$

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(s)ds = \vec{v}(0) + \int_0^t \langle \sin(s), s, 2 \rangle ds$$

$$= \vec{v}(0) + \left[\langle -\cos(s), \frac{s^2}{2}, 2s \rangle \right]_0^t$$

$$= \vec{v}(0) + \langle -\cos(t), \frac{t^2}{2}, 2t \rangle - \langle -\cos(0), \frac{0^2}{2}, 2\cdot 0 \rangle$$

$$= \vec{v}(0) + \langle -\cos(t), \frac{t^2}{2}, 2t \rangle - \langle -1, 0, 0 \rangle$$

$$= \vec{v}(0) + \langle -\cos(t), \frac{t^2}{2}, 2t \rangle$$

not always zero

Motion in Space:

position $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

velocity $\vec{v}(t) = \vec{r}\,'(t) = \langle x', y', z' \rangle$

acceleration $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$

$= \langle x'', y'', z'' \rangle$

Example: object's position:

$$\vec{r}(t) = \langle 1, t^2, e^{-t} \rangle$$

Q: What is the speed of object

speed: $|\vec{v}(t)| = |\vec{r}\,'(t)| = |\langle 0, 2t, -e^{-t} \rangle|$

$$= \sqrt{0^2 + (2t)^2 + (-e^{-t})^2}$$

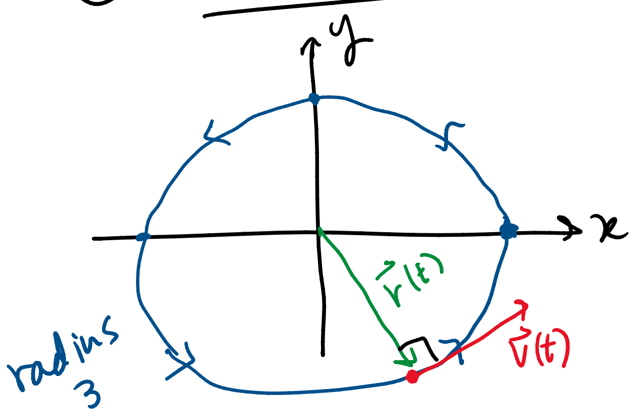
$$\boxed{= \sqrt{4t^2 + e^{-2t}}}$$ scalar

Two Common Types of Motion

① straight line motion

$$\vec{r}(t) = \langle x_0 + at,\ \underbrace{y_0 + bt}_{\text{eqn of a line}},\ z_0 + ct \rangle$$

② circular motion : – motion lies on a circle



$$\vec{r}(t) = \langle 3\cos(t),\ 3\sin(t) \rangle$$

Note:
$$|\vec{r}(t)| = \sqrt{(3\cos t)^2 + (3\sin t)^2} = 3$$
constant for all t

$$\vec{v}(t) = \langle -3\sin(t),\ 3\cos(t) \rangle$$
$$\vec{a}(t) = \langle -3\cos(t),\ -3\sin(t) \rangle$$
$$= -\vec{r}(t)$$

Note: $\vec{r}(t) \cdot \vec{v}(t) = \langle 3\cos(t), 3\sin(t) \rangle \cdot \langle -3\sin(t), 3\cos(t) \rangle$
$$= -9\sin(t)\cos(t) + 9\sin(t)\cos(t) = 0$$

$\vec{r}(t) \perp \vec{v}(t)$ for all t

$$\boxed{\text{Circular Motion:}\quad |\vec{r}(t)| = \text{constant},\quad \vec{r}(t)\cdot\vec{v}(t) = 0}$$

Note: $\vec{r}(t) = \langle 3\cos(2t),\ 3\sin(2t) \rangle$

$$\vec{a} = -4\vec{r}(t) = \vec{r}\,''(t)$$

Projectile Motion:

We want to find the height and distance an object travels when it is thrown in the air

<u>We know</u>: $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$

acceleration of gravity $\rightarrow$ constant $-g$ in the vertical direction

$$g = 9.8\, m/s^2 = 32\, ft/s^2$$

$\vec{a}(t) = \langle 0, -g \rangle$

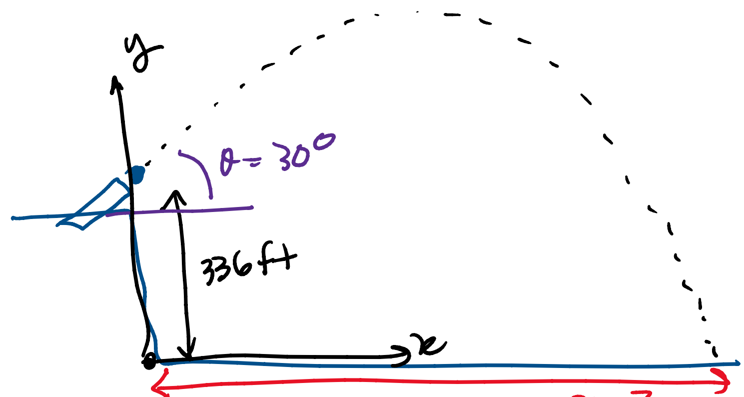
initial position: $\vec{r}_0 = \langle x_0, y_0 \rangle$

initial velocity: $\vec{v}_0 = \langle u_0, v_0 \rangle$

solve for $\vec{v}(t)$, $\vec{r}(t)$

<u>Ex</u>:

A cannonball is shot off a cliff at an angle $\theta = 30°$ from horizontal with speed 640 ft/s



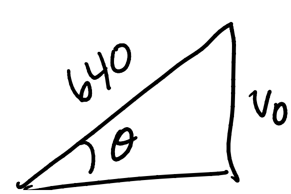
<u>Q</u>: How far does the cannonball go?

$a(t) = \langle 0, -32 \rangle$ $\qquad |\vec{v}_0| = 640 \qquad \theta = 30$

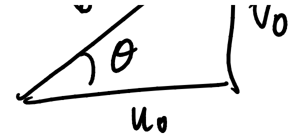
$\vec{r}_0 = \langle 0, 336 \rangle$

$\vec{v}_0 = \langle |v_0|\cos\theta, |v_0|\sin\theta \rangle = \langle 640\frac{\sqrt{3}}{2}, 640 \cdot \frac{1}{2} \rangle$

$= \langle 320\sqrt{3}, 320 \rangle$



$= \langle 320\sqrt{3}, 320 \rangle$



<u>velocity:</u>

$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 0, -32 \rangle\,dt$

$= \langle c_1, -3\cancelto{0}{2}t + c_2 \rangle$

$V_0 = \langle 320\sqrt{3}, 320 \rangle = \vec{v}(t=0) = \langle c_1, c_2 \rangle$

$$\boxed{\vec{v}(t) = \langle 320\sqrt{3}, -32t + 320 \rangle}$$

<u>position:</u>

$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \langle 320\sqrt{3}, -32t + 320 \rangle\,dt$

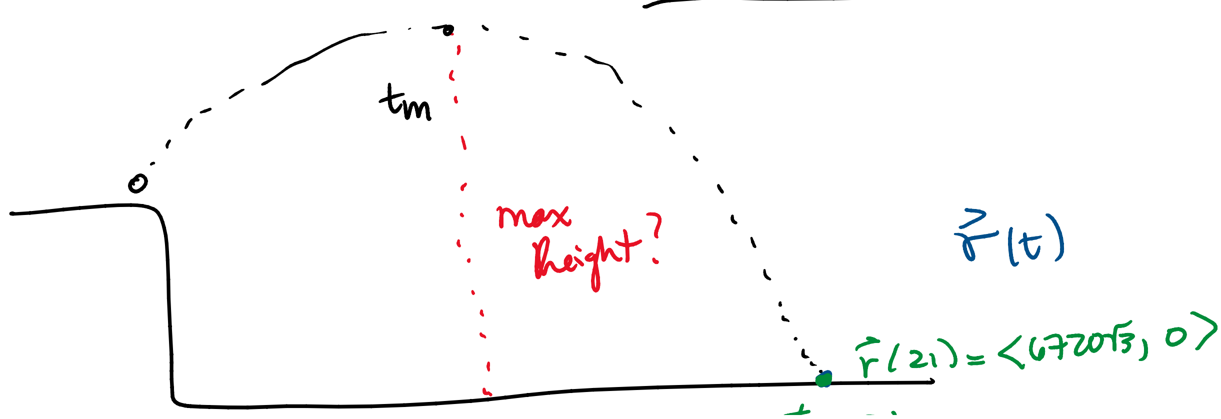
$= \langle 320\sqrt{3}t + D_1, -16t^2 + 320t + D_2 \rangle$

$\vec{r}(0) = \langle 0, 336 \rangle = \vec{r}(t=0) = \langle D_1, D_2 \rangle$

$$\boxed{\vec{r}(t) = \langle 320\sqrt{3}t, -16t^2 + 320t + 336 \rangle}$$

$\boxed{\text{time of flight} \quad t = 21}$

distance travelled $x(t=21) = 320\sqrt{3}(21)$ $\boxed{= 6720\sqrt{3}\text{ ft} \approx 11,639.4\text{ ft}}$



$\underline{Q}$: What is the maximum height?

slope = 0

$v(t) = y'(t) = 0$

$-32t + 320 = 0$

$\boxed{t_{max} = 10s}$

height: $y(t_{max}) = \cancel{320\sqrt{3} \cdot 10}$

$\boxed{\cancel{= 3200\sqrt{3}} \quad \cancel{\approx 5,542.6} \text{ ft}}$

$$\text{height} = -16t^2 + 320t + 336 \Big|_{t=10}$$

$$= -1600 + 3200 + 336$$

$$\approx 1936$$

$\underline{\text{same problem:}}$

There is a wind in negative x-direction acceleration $-10 \text{ft}/s^2$

$\vec{v}_0 = \langle 320\sqrt{3}, 320 \rangle$

$\vec{r}_0 = \langle 0, 336 \rangle$

$\vec{a}(t) = \langle -10, -32 \rangle$

$\vec{v}(t) = \langle -10t + 320\sqrt{3}, -32t + 320 \rangle$

$\vec{r}(t) = \langle -5t^2 + 320\sqrt{3}t, \underbrace{-16t^2 + 320t + 336}_{t_f = 21} \rangle$

distance travelled:

$$-5(21)^2 + 320\sqrt{3} \cdot 21 = \boxed{9434.4\text{ft}}$$