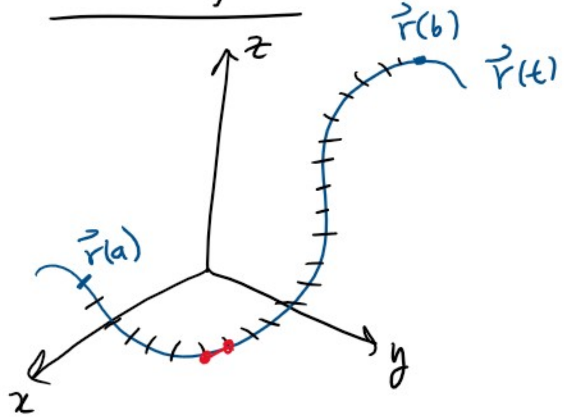
Lesson 8 - 9/8/2023 - 4:30pm

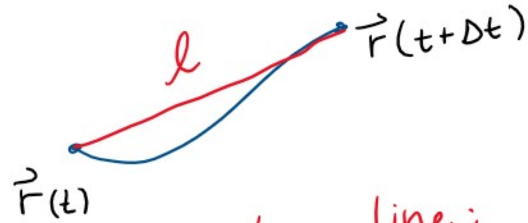
14.4: Length of Curves

14.5: Curvature

Arclength :



Q: What is length from $\vec{r}(a)$ to $\vec{r}(b)$ along $\vec{r}(t)$?



approximate by a line:

$$\ell = |\vec{r}(t+\Delta t) - \vec{r}(t)|$$

To get the total arclength

$$L = \lim_{\Delta t \to 0} \sum_{i=a}^{b} |\vec{r}(t_i + \Delta t) - \vec{r}(t_i)|$$

$$= \lim_{\Delta t \to 0} \sum_{i} \left| \frac{\vec{r}(t_i+\Delta t) - \vec{r}(t_i)}{\Delta t} \right| \Delta t$$

$$\boxed{\begin{aligned} L &= \int_a^b |\vec{r}\,'(t)|\,dt \qquad \text{Arclength} \\ &= \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\,dt \end{aligned}}$$

Idea: $\int_a^b \text{speed } dt = \text{distance from } a \text{ to } b$

Arc length Parameterization:

Define the arc-length function $s(t)$

$$s(t) = \int_a^t |\vec{r}\,'(u)|\,du$$

dummy variable u

indep var t

indep var t

$s(t) = \int_a$

$s(t)$ represents the arclength of $r(t)$ on $[a, t]$

Ex: $\vec{r}(t) = \langle \cos(t), \sin(t), 3t \rangle$ let $a=0$

$|\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 3^2} = \sqrt{10}$

$s(t) = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t \sqrt{10}\,du = \sqrt{10}\,t$

$$\boxed{s = \sqrt{10}\,t}$$

Relationship between arclength s and time t

$$t = \frac{s}{\sqrt{10}}$$

Reparameterize $\vec{r}(t)$:

$\tilde{\vec{r}}(s) = \vec{r}\left(\frac{s}{\sqrt{10}}\right) = \left\langle \cos\left(\frac{s}{\sqrt{10}}\right), \sin\left(\frac{s}{\sqrt{10}}\right), \frac{3s}{\sqrt{10}} \right\rangle$

this is a function of arclength

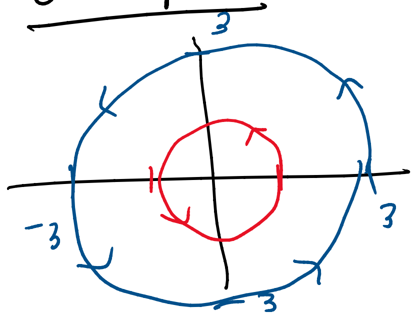
Thm: If $|\vec{r}\,'(t)| = 1$, then $\vec{r}(t)$ is parameterized by arclength

$s = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t 1\,du = t$

Curvature:

Q: At a point t, how "sharply" does a curve $\vec{r}(t)$ turn? $\longrightarrow$ curvature

Example:

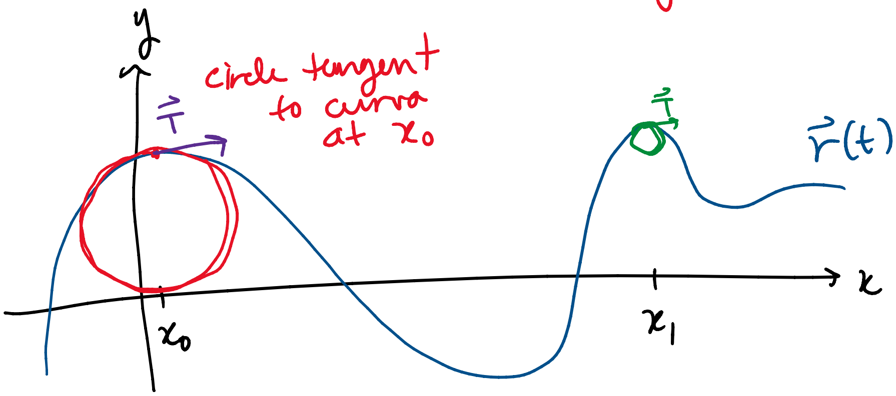


$\vec{r}(t) = \langle 3\cos t, 3\sin t \rangle$

a circle has constant curvature

$\vec{r}(t) = \langle \cos t, \sin(t) \rangle$

a smaller circle has greater curvature



Def: Let $\vec{r}(s)$ describe a curve where s is the arclength parameter. Then the **curvature** κ at s is:

$$\kappa(s) = \left| \frac{d\vec{T}}{ds} \right| = |\vec{T}'(s)|$$

← rate of change of the unit tangent vector at s

$$\vec{T}(s) = \frac{\vec{r}'(s)}{\cancel{|\vec{r}'(s)|} = 1} = \vec{r}'(s)$$

when s is the arclength

$\vec{T}(s) = \vec{r}'(s)$

$\longrightarrow \kappa = |\vec{T}'(s)| = |\vec{r}''(s)|$

Ex: $\vec{r}(s) = \langle 3\cos(\frac{s}{3}), 3\sin(\frac{s}{3}) \rangle$ - parametrized by arclength

$$\kappa = |\vec{T}'(s)| = \left| \frac{d}{ds} \vec{T}(s) \right| = \left| \frac{d}{ds} \vec{r}'(s) \right|$$

$$= \left| \frac{d}{ds} \left\langle -\cancel{3} \cdot \cancel{\tfrac{1}{3}} \sin(\tfrac{s}{3}), \cancel{3} \cdot \cancel{\tfrac{1}{3}} \cos(\tfrac{s}{3}) \right\rangle \right|$$

$$= \left| \left\langle -\tfrac{1}{3}\cos(\tfrac{s}{3}), -\tfrac{1}{3}\sin(\tfrac{s}{3}) \right\rangle \right|$$

for

$$= \left| \left\langle -\tfrac{1}{3}\cos\left(\tfrac{s}{3}\right), -\tfrac{1}{3}\sin\left(\tfrac{s}{3}\right) \right\rangle \right|$$

$$= \sqrt{\left(-\tfrac{1}{3}\cos\left(\tfrac{s}{3}\right)\right)^2 + \left(-\tfrac{1}{3}\sin\left(\tfrac{s}{3}\right)\right)^2} = \boxed{\frac{1}{3} = K}$$ for all s

NOTE: For circle, $K = \dfrac{1}{R}$ where R is the radius

Q: What if we are only given $\vec{r}(t)$ as a fcn of time t and not arclength s?

Use Chain Rule:

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt}$$

Alternate for curvature

$$\boxed{K = \left|\frac{d\vec{T}}{ds}\right| = \frac{\left|\frac{d\vec{T}}{dt}\right|}{\left|\frac{ds}{dt}\right|} = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} \; *}$$

$$* \quad \frac{ds}{dt} = \frac{d}{dt}\int_0^t |\vec{r}'(u)|\,du = |\vec{r}'(t)|$$

Alternate Formula:

$$\boxed{K = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}}$$

← derived on page 904–905 in textbook

Ex: Find the curvature $K(0)$ at $t=0$

$$\vec{r}(t) = \langle \sqrt{2}\,t, e^t, e^{-t} \rangle$$

$$|\vec{r}'(t)| = |\langle \sqrt{2}, e^t, -e^{-t} \rangle|$$

$$= \sqrt{2 + e^{2t} + e^{-2t}}$$

$$= \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t} = |\vec{v}|$$

$$= \sqrt{(e^t+e^{-t})^2} = e^t + e^{-t} = |\vec{v}|$$

use: $K = \dfrac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3}$ $\quad \vec{v} = \langle \sqrt{2}, e^t, -e^{-t} \rangle$

$\vec{a} = \langle 0, e^t, e^{-t} \rangle$

$$|\vec{v}\times\vec{a}| = \left| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ \sqrt{2} & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{matrix} \right| = \left| \langle 2, -\sqrt{2}e^{-t}, \sqrt{2}e^{t} \rangle \right|$$

$$= \sqrt{4 + 2e^{-2t} + 2e^{2t}}$$

$$= \sqrt{2}(e^t + e^{-t})$$

$$K = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} = \frac{\sqrt{2}\cancel{(e^t+e^{-t})}}{(e^t+e^{-t})^{\cancel{3}\,2}}\Bigg|_{t=0} = \frac{\sqrt{2}}{(2)^2} = \boxed{\frac{\sqrt{2}}{4} = K(0)}$$